Game Theory

Lecture Notes By

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March 2007

Chapter 1: Introduction to Game Theory

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* hari@csa.iisc.ernet.in. *A more thorough version would be available soon in this space.*

- According to Myerson, *game theory* may be defined as the study of mathematical models of *conflict* and *cooperation* between rational, intelligent decision makers.
- Game theory provides general mathematical techniques for analyzing situations in which two or more individuals (called players or agents) make decisions that influence one another's welfare.
- More appropriate phrases that describe game theory are:
  - Conflict Analysis
  - Interactive Decision Theory
- According to Osborne and Rubinstein, a game is a description of strategic interaction that occurs among decision-making entities.

A game can be described by four elements (Mascollel, Whinston, Green):

1. **Players** (also called agents): an individual or a group of individuals making a decision
2. **Rules** specify
   - who moves when?
   - what can they do?
   - what do they know when they move?
3. **Outcomes**: What happens when each player plays in a particular way?
4. **Preferences**: What are the players' preferences over the possible outcomes.

Example 1: Matching Pennies

(1) Players: {1,2}

(2) Rules: Each player simultaneously puts down a rupee coin, heads up or tails up

(3) Outcomes: {(H,H), (H,T), (T,H), (T,T)}

(4) Preferences: If the two coins match, player 1 pays Rs. 100 to player 2; otherwise player 2 pays Rs. 100 to player 1.

- Let us call this Matching Pennies: Version A.

Example 2: Tick-Tack-Toe

(1) Players {X, O}

(2) Rules

  - There is a board that consists of 9 squares in 3 rows and 3 columns
  - Players take turns putting their marks (either X or O) into an unmarked square.
  - Player X moves first
  - The players observe all the choices made by the players so far.

(3) Outcomes: Win, loss, or draw

  - The first player to have 3 marks in a horizontal row or a vertical column or along the diagonal wins and the other one loses
  - If no one succeeds even after all 9 squares are marked, it is a draw.

(4) Preferences

  - The player who wins will receive Rs. 100 from the player who loses
  - No payments are made if it is a draw.

Implicit Assumptions in Game Theory

(1) The players are assumed to be *rational*.

  - A decision maker is said to be **rational** if the agent makes decisions consistently in pursuit of his own objectives.
  - It is assumed that each player's objective is to maximize the expected value of his own payoff measured in some utility scale (This is the *selfish* part of rationality)
  - The above notion of rationality (maximization of expected utility payoff) is attributed to
    * Bernoulli (1738)
    * von Neumann and Morgenstern (1947)

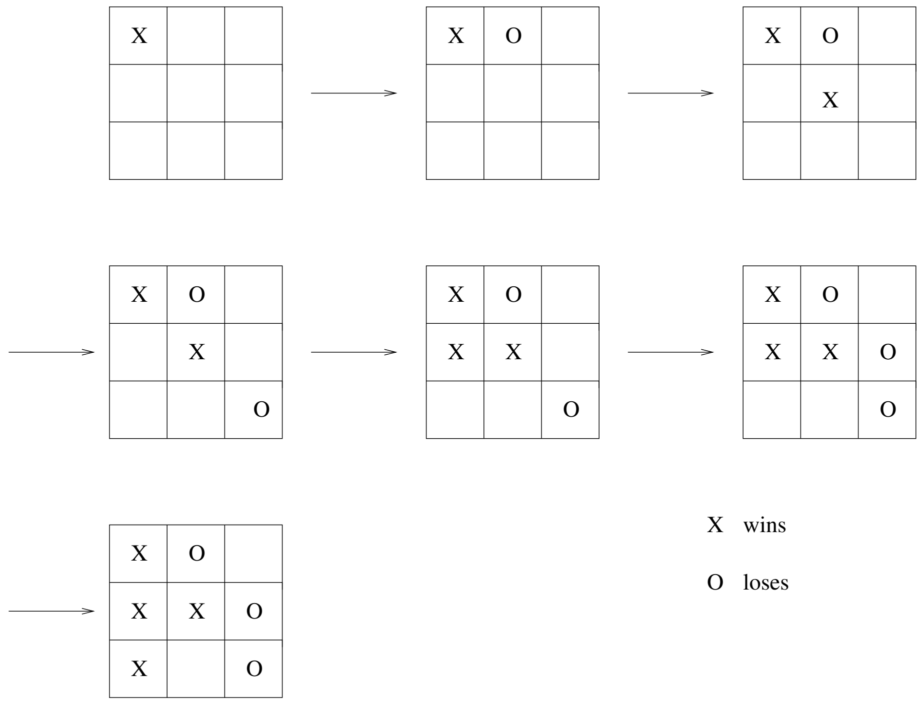

Figure 1: A sequence of moves in Tick-Tack-Toe

- von Neumann and Morgenstern stated and proved the *Expected Utility Maximization Theorem* which, under quite weak assumptions about how a rational decision maker should behave, shows for any rational decision maker that there must exist some way of assigning utility numbers to different outcomes in a way that he would always choose the option that maximizes his expected utility.
- A key assumption for the expected utility maximization theorem is the *sure-thing* or *substitution* axiom. This axiom informally means the following: If a decision maker prefers option 1 over option 2 when event A occurs and he would prefer option 1 over option 2 when event A does not occur, then he should prefer option 1 over option 2 even before he learns whether event A will occur or not.
- Using the above assumption and certain technical regularity conditions, von Neumann and Morgenstern showed that there exists some utility scale such that the decision maker always prefers the options that give him the highest expected value.

  Note: Maximizing expected utility payoff is not necessarily the same as maximizing expected monetary payoff. In general they are non-linearly related. Specifically, it depends on the nature of the agent: risk-loving or risk-averse or risk-neutral.
- When there are two or more decision makers, it is often the case that the rational solution to each individual's decision problem depends on the others' individual problems and vice-versa. None of the problems may be solvable without understanding the solutions of the other problems.
  * Thus when rational decision makers interact, their decision problems must be analyzed together, like a system of simultaneous equations.

- Such analysis is the subject of game theory.

  Note: *Selfishness or self-interestedness* is an important implication of rationality.

(2) The players are assumed to be *intelligent*.

  - This means that each player in the game knows everything about the game that a game theorist knows and he can make any inferences about the game that a game theorist can make.
  - In particular, an intelligent player is *strategic*, that is, he takes into account his knowledge or expectation of behavior of other agents. He is capable of doing the required computations.

Note: Justifying Rationality and Intelligence: Myerson provides the following explanation to show that the assumptions of rationality and intelligence are reasonable.

- The assumption that all individuals are rational and intelligent may not be satisfied in the strict sense in a typical real-world situation.
- However, theories that are not consistent with the assumptions or rationality and intelligence are not likely to be sound, for,
  - if a theory predicts that some individuals will be systematically fooled or led into making mistakes, then such a theory will tend to lose validity when individuals learn to better understand the situations.
- Thus rationality and intelligence are very reasonable assumptions.

Common Knowledge

- This is an important implication of *intelligence*.
- Aumann (1976) defined *common knowledge* as follows: A fact is common knowledge among the players if every player knows it, every player knows that every player knows it, and so on. That is, every statement of the form "((every player knows that) to the power k) every player knows it" is true for $k = 0, 1, 2, \ldots$.
- A player's *private information* is any information that he has that is not common knowledge among all the players.
- The intelligence assumption means that whatever we may know or understand about the game must be known or understood by the players of the game.
- Thus whatever is the model of the game that is being studied is also known to the players.
  - Since the players all know the model, being intelligent, they also know that they all know the model.
  - Thus the model is common knowledge

An Example for Illustrating Common Knowledge:

- There are 5 mothers A B C D E
  and 5 sons a b c d e

- Everyday the 5 mothers meet and if they think their boy is well behaved, they will praise the virtues of their respective boys. If a mother knows that her son is not well behaved (for example, tells lies), then she will cry.

- It turns out that all the boys have the habit of telling lies and they all know it and each boy informs his own mother that the other boys are bad boys. Thus A knows that b, c, d, e are bad boys but does not know that her own son a is also a bad boy.

- Since each mother does not know that her boy is badly behaved, it turns out that every mother keeps praising her son everyday.

- One fine day, the class teacher meets all the mothers and tells them one of the boys is a bad boy.

- Thereafter, on the next day, all of them praise their boy; the same happens on the 2^{nd} day, 3^{rd}, and the 4^{th} day.

- On the 5^{th} day, all the mothers cry simultaneously.

- Note that the announcement made by the class teacher is common knowledge and that is what makes all the mothers cry on the fifth day.

Bounded Rationality: Osborne and Rubinstein provide the following explanation regarding the important notion of bounded rationality.

- Game theory, in its current form, assumes that all players have identical capabilities of perception and computation. It does not model asymmetries in abilities or perceptions of situations.

- For example, the game of chess, when analyzed using game theory can be solved using an algorithm. This was shown by Zermelo in 1913. Thus chess becomes a trivial game for rational players.

- Zermelo's Result: Zermelo showed that the game of chess has a unique "equilibrium" outcome with the property that a player who follows the suggested strategy can be sure that the outcome will be at least as good as the equilibrium outcome:
  - However only the existence of the equilibrium outcome is shown
  - It is yet to be computed

- Thus a game theoretic model of chess reveals an important fact about the game and suggests that it is not at all interesting for rational players.

- However the game theoretic model does not capture the asymmetric abilities of the players which is what makes it interesting.

Classification of Games

1. Non-cooperative games and cooperative games

- Non-cooperative games are those in which the possible actions of individual players are the primitives; in cooperative games, joint actions of groups of players are the primitives.
- John Harsanyi (1966) explained that a game is cooperative if commitments (agreements, promises, threats) are enforceable and that a game becomes non-cooperative if the commitments are not enforceable.

2. Strategic form games and extensive form games

    - A strategic game (also called simultaneous move game or normal from game) is a model or a situation where each player chooses his plan of action once and for all and all players exercise their decision simultaneously.
    - An extensive form game specifies a possible order of events and each player can consider his plan of action whenever a decision has to be made.

3. Games with perfect information and games with imperfect Information

    - When the players are fully informed about each other's moves (each player, before making a move, knows the past moves of all other players as well as his own past moves), the game is said to be of perfect information.
    - Otherwise the game is said to be with imperfect information.

4. Games with complete information and games with incomplete information

    A game with incomplete information is one in which, at the first point in time when the players can begin to plan their moves, some players already have private information about the game that other players do not know. In a game with complete information, every aspect of the game is common knowledge.

    There are many other categories of games: (More about this later.)

    - Dynamic games
    - Repeated games
    - Games with communication
    - Multi-level games (Stackelberg games)
    - Differential games
    - Stochastic games