Game Theory

Lecture Notes By

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Chapter 4: Dominant Strategy Equilibria

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Strictly Dominated Strategy

Given a game $G = (N, (S_i), (u_i))$, a strategy $s_i \in S_i$ is said to be *strictly dominated* if there exists another strategy $s_i' \in S_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

In such a case, we say strategy s_i' strictly dominates strategy s_i.

Strictly Dominant Strategy

A strategy $s_i^* \in S_i$ is said to be a strictly dominant strategy for player i if it strictly dominates every other strategy $s_i \in S_i$. That is, $\forall s_i \neq s_i^*$,

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

Strictly Dominant Strategy Equilibrium

A profile of strategies $(s_1^*, s_2^*, \ldots, s_n^*)$ is called a strictly dominant strategy equilibrium of game $\Gamma =< N, (S_i), (u_i) >$ if $\forall i = 1, 2, \ldots, n$, s_i^* is a strictly dominating strategy for player i.

Example: Prisoner's Dilemma

Recall the prisoner's dilemma problem where $N = \{1, 2\}$ and $S_1 = S_2 = \{C, NC\}$ and the payoff matrix is given by:

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Observation 1: NC is strictly dominated by C for player 1:

$$\begin{aligned} u_1(C, NC) &> u_1(NC, NC) \\ u_1(C, C) &> u_1(NC, C) \end{aligned}$$

Observation 2: NC is strictly dominated by C for player 2:

$$\begin{aligned} u_2(NC, C) &> u_2(NC, NC) \\ u_2(C, C) &> u_2(C, NC) \end{aligned}$$

Thus C is a strictly dominant strategy for player 1 and also for player 2. Therefore (C, C) is a strongly dominant strategy equilibrium for the PD game.

Note: If a (rational) player has a strictly dominating strategy then we should expect him to play it. On the other hand, if a player has a strictly dominated strategy, then we should expect him not to play it.

Weak Dominance

Given a game $\Gamma = < N, (S_i), (u_i) >$, a strategy $s_i \in S_i$ is said to be weakly dominated by a strategy $s'_i \in S_i$ for player i if for all $s_i \in S_i$,

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \ \ \forall s_{-i} \in S_{-i}$$

with strict inequality satisfied for at least one s_{-i}. The strategy s'_i is said to weakly dominate strategy s_i.

Weakly Dominant Strategy

A strategy s^*_i is said to be a weakly dominant strategy for player i if it weakly dominates every other strategy $s_i \in S_i$.

Weakly Dominant Strategy Equilibrium

Given a game $\Gamma = < N, (S_i), (u_i) >$, a strategy profile $(s^*_1, \ldots, s^*_n)$ is called a weakly dominant strategy equilibrium if for $i = 1, \ldots, n$, the strategy s^*_i is a weakly dominant strategy for player i.

Example: Modified Prisoner's Dilemma

Consider the following payoff matrix of a slightly modified version of the prisoner's dilemma problem.

	2	
1	NC	C
NC	$-2, -2$	$-10, -2$
C	$-2, -10$	$-5, -5$

Observation 1: C is weakly dominant for player 1.

$$\begin{aligned} u_1(C, NC) &\geq u_1(NC, NC) \\ u_1(C, C) &> u_1(NC, C) \end{aligned}$$

Observation 2: C is weakly dominant for player 2.

$$\begin{aligned} u_2(NC, C) &\geq u_2(NC, NC) \\ u_2(C, C) &> u_2(C, NC) \end{aligned}$$

Therefore the strategy profile (C, C) is a weakly dominant strategy equilibrium.

Example: Second Price Sealed Bid Auction

Consider the second price sealed bid auction for selling a single indivisible item. This is also famously called the *Vickrey Auction*. Recall that there are n bidders: $N = \{1, 2, \ldots, n\}$. The valuations that the players attach to the item are respectively, $v_1, v_2, \ldots, v_n$. Let $b_1, b_2, \ldots, b_n$ be the bids and $b = (b_1, b_2, \ldots, b_n)$. Assume that $b_i \in (0, \infty)$ for $i = 1, 2, \ldots, n$. Assume also that the item is awarded to the player who has the lowest index among all the highest bidders. The allocation function is formally expressed as:

$$\begin{aligned} x_i(b_1, \ldots, b_n) \quad &= 1 \text{ if } \quad b_i > b_j \text{ for } j = 1, 2, \ldots, i-1 \text{ and} \\ & \qquad\qquad b_i \geq b_j \text{ for } j = i+1, \ldots, n \\ &= 0 \text{ else} \end{aligned}$$

The payoff for each bidder is given by:

$$v_i(b_1, \ldots, b_n) = x_i(b_1, \ldots, b_n)(v_i - p_i)$$

where p_i is the amount paid by the winning bidder. Being second price auction, the winner pays only the next highest bid. Note that $< N, (S_i)_{i \in N}, (u_i)_{i \in N} >$ is a normal form game. We now show that the strategy profile $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ is a weakly dominant strategy equilibrium for this game.

Proof

Consider player 1. His value is v_1 and bid is b_1. The other players have bids $b_2, \ldots, b_n$ and valuations $v_2, \ldots, v_n$. We consider the following cases.

1. **Case 1**: $v_1 \geq \max(b_2, \ldots, b_n)$. There are two sub-cases here:
   (a) $b_1 \geq \max(b_2, \ldots, b_n)$
   (b) $b_1 < \max(b_2, \ldots, b_n)$
2. **Case 2**: $v_1 < \max(b_2, \ldots, b_n)$. There are two sub-cases here:
   (a) $b_1 \geq \max(b_2, \ldots, b_n)$
   (b) $b_1 < \max(b_2, \ldots, b_n)$

We analyze these cases separately below.

Case 1

- $v_1 \geq \max(b_2, \ldots, b_n)$ and $b_1 \geq \max(b_2, \ldots, b_n)$ imply that bidder 1 is the winner. This implies that$u_1 = v_1 - \max(b_2, \ldots, b_n) \geq 0$.

- $v_1 \geq \max(b_2, \ldots, b_n)$ and $b_1 < \max(b_2, \ldots, b_n)$ imply that 1 is not the winner. This implies that $v_1 = 0$

- If $b_1 = v_1$, then since $v_1 \geq \max(b_2, \ldots, b_n)$,
$$u_1 = v_1 - \max(b_2, \ldots, b_n)$$
Therefore, if $b_1 = v_1$, the utility u_1 is $\geq$ maximum utility obtainable.

- Thus $b_1 = v_1$ is a weakly dominant strategy for a player 1.

Case 2

- $v_1 < \max(b_2, \ldots, b_n)$ and $b_1 \geq \max(b_2, \ldots, b_n)$ imply that 1 is the winner.
$$u_1 = v_1 - \max(b_2, \ldots, b_n) \leq 0$$

- $v_1 > \max(b_2, \ldots, b_n)$ and $b_1 < \max(b_2, \ldots, b_n)$ imply that 1 is not the winner. Therefore $u_1 = 0$

- If $b_1 = v_1$, then b_1 being less than $\max(b_2, \ldots, b_n)$, 1 is not the winner. Therefore $u_1 = 0$.

From the above analysis, we have that
$$u_1(v_1, b_2, \ldots, b_n) \geq u_1(\hat{b_1}, b_2, \ldots, b_n) \;\; \forall \hat{b_1} \in S_1 \quad \forall b_2 \in S_2 \quad \ldots, \quad b_n \in S_n$$
Thus $b_1 = v_1$ is a weakly dominant strategy for a player 1. Similarly, $b_i = v_i$ is a weakly dominant strategy for player i where $i = 2, 3, \ldots, n$. Therefore $(v_1, \ldots, v_n)$ is a weakly dominant strategy equilibrium.

Iterated Elimination of Strictly Dominated Strategies

We can eliminate strictly dominated strategies as possible choices.

- Iterated elimination of strictly dominated strategies requires rationality and intelligence assumptions.

- With each elimination of strategies, it becomes possible for additional strategies to become strictly dominated.

- Note that each additional iteration requires that players' knowledge of each others' rationality be one level deeper.

- If several strategies are strictly dominated, then we can eliminate them all at once or any sequence without changing the set of strategies that we end up with.

- Such iterated elimination may not lead to a unique prediction for the game but will lead to a reduced form of the game that is easier to analyze.
  - In some cases, we might still get a unique prediction using rationality and intelligence assumptions.

Example: DA's Brother

Consider the following payoff matrix of another variation of the Prisoner's dilemma problem.

	2	
1	NC	C
NC	$0, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

- Note that NC and C are not strictly dominated for player 1.
- Note that NC is strictly dominated for player 2. So, we eliminate NC for player 2. To eliminate this, player 2 should know that P, is rational.
- Now we are left with only the second column. Now we can zero in on (C, C) strategy profile by looking at the second column.

Example

	2	
1	L	R
U	5,1	4,0
M	7,1	5,0
D	6,4	4,4

Player 1: U is strictly dominated by M. D is also strictly dominated by M. Thus both U and D can be eliminated. This leaves us with only the second row.

Player 2: R is strictly dominated by L. So, we can eliminate second column.
This gives uniquely the profile (M, L) which is a strongly dominant strategy equilibrium.

Example: Cournot Duopoly Game

This is one example of a game for which the iterated removal of strictly dominated strategies yields a unique prediction.

Example

	2	
1	L	R
U	5,1	4,0
M	6,0	3,1
D	6,4	4,4

- **Player 1**: U and M are both weakly dominated by strategy D. Therefore D is a weakly dominant strategy of player 1.

- **Player 2**: Neither L weakly dominates R not vice-versa.

Note: Unlike in the case of strictly dominated strategies, weakly dominated strategies cannot be simply eliminated based solely on principles of rationality and intelligence.

Example

We have seen that U and M are weakly dominated. Can we then eliminate M? We cannot eliminate M if we know for sure that player 2 will play only L (i.e., will not play R). So, M can only be eliminated if we are sure that player 2 will play R with probability 1.

Note: We can only delete a weakly dominated strategy only if we are clear that every strategy combination of the other players occurs with positive probability.

Note: Iterated deletion of weakly dominated strategies depends on the order of deletion.

Example

	2	
1	L	R
U	5,1	4,0
M	6,9	3,1
D	6,4	4,4

Player 1: U and M are weakly dominated by D for player 1. Also R is dominated weakly by L for player 2.

- If we first eliminate U and then eliminate R, we are led to the outcome (M,L).
- If we first eliminate M and then eliminate R, we are led to the outcome (D,L).

A General Procedure for Iterative Elimination of Strongly Dominated Strategies

This procedure is taken from the book by Myerson. Start with the game

$$G = (N, (S_i), (u_i))$$

For any player i, let $S_i^{(1)}$ denote the set of all strategies in S_i that are not strongly dominated for it. Consider the game

$$G^{(1)} = (N, (S_i^{(1)}), (u_i))$$

where each u_i function is actually the restriction of the original utility function to the new smaller domain.

By induction, for every positive integer k, we can define the strategic form game $G^{(k)}$:

$$G^{(k)} = (N, (S_i^{(k)}), (u_i))$$

where, for each player $i \in N$, $C_i^{(k)}$ is the set of all strategies in $C_i^{(k-1)}$ that are not strongly dominated for player i in the game $G^{(k-1)}$.

Since G is a finite game, there must exist some number K such that

$$S_i^{(k)} = S_i^{(k+1)} = S_i^{(k+2)} = \dots$$

Let $G^{(\infty)} = G^{(k)}$ and $S_i^{(\infty)} = S_i^{(k)}$ for every $i \in N$.

The strategies in $S_i^{(\infty)}$ are *iteratively undominated* in the strong sense for player i. The Game $G^{(\infty)}$ is called the *residual game* generated from G by iterative strong domination.

- Since the players are rational, it can be concluded that no player would use any strongly dominated strategy. Thus each player i must be expected to choose a strategy in $S_i^{(1)}$.

- Since all players are intelligent, they should know that no player i will use a strategy outside of $S_i^{(1)}$. Since $S_i^{(2)}$ is the set of all strategies that are best responses to probability distributions over

  $$X_{j \in N \setminus \{i\}} S_i^{(1)}$$

  each player i must choose a strategy in $S_i^{(2)}$.

- Since each player i is intelligent, he must also know that every player j will use a strategy in $S_i^{(2)}$ and so player i must use a strategy in $S_i^{(3)}$.

- Using the assumptions of rationality and intelligence in this way repeatedly, we can conclude that each player i must use a strategy in $S_i^{(\infty)}$.