Game Theory

Lecture Notes By

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Chapter 5: Pure Strategy Nash Equilibrium

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Nash Equilibrium

John Nash (who won the Nobel Prize in 1994) is credited with perhaps the most celebrated notion in game theory. This notion was proposed in the following two papers:

- John F. Nash, Noncooperative games. Annals of Mathematics, Volume 54, pages 289-95, 1951.
- John F. Nash, Equilibrium points in N-person games. Proceedings of the National Academy of Sciences of USA, volume 36, pages 48-49, 1950.

Dominant strategy equilibria (strongly dominant, weakly dominant), if they exist, are very nice and desirable but rarely do they exist because they are too demanding. In a two player game, a dominant strategy equilibrium requires that each player's choice be optimal against all choices of the other player. If we only insist that is optimal for the optimal choices of the other player, we get a Nash equilibrium.

Definition

Given a game $\Gamma = (N, (S_i), (u_i))$ with pure strategies, the strategy profile

$$s^* = (s_1^*, s_2^*, \ldots, s_n^*)$$

is said to be a pure strategy Nash equilibrium of Γ if,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i, \quad \forall i = 1, 2, \ldots, n.$$

That is, each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players.

Recall the notation

$$\begin{aligned}
\Gamma &= (N, (S_i), (u_i)) \\
N &= \{1, 2, \ldots, n\} \\
S &= S_1 \times S_2 \times \cdots \times S_n \\
s &= (s_1, s_2, \ldots, s_n) \in S \\
S_{-i} &= S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \\
s_{-i} &= (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \\
u_i &: S_1 \times \cdots \times S_n \to \mathbb{R}
\end{aligned}$$

Best Response Correspondence for player i: Given a game $\Gamma = (N, (S_i), (u_i))$, the best response correspondence for player i is the mapping

$$B_i : S_{-i} \to 2^{S_i}$$

defined by

$$B_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \quad \forall s_i' \in S_i\}$$

That is, given a profile s_{-i} of strategies of the other players, $B_i(s_{-i})$ gives the set of all best response strategies of player i.

Alternative Definition of Nash Equilibrium

Given a pure strategy game $\Gamma = (N, (S_i), (u_i))$, the strategy profile $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium iff,

$$s_i^* \in B_i(s_{-i}^*), \quad \forall i = 1, \ldots, n$$

Note: For $(s_1^*, \ldots, s_n^*)$ to be a Nash equilibrium, it must be that no player can profitably deviate from his Nash equilibrium strategy given that the other players are playing their Nash equilibrium strategies.

Example 1: The BOS Game

	2	
1	M	F
M	2,1	0, 0
F	0,0	1,2

There are two Nash equilibria here, namely (M,M) and (F,F). The profile (M,M) is Nash equilibrium because

$$\begin{aligned}
u_1(M, M) &> u_1(F, M) \\
u_2(M, M) &> u_2(M, F)
\end{aligned}$$

The profile (F, F) is a Nash equilibrium because

$$\begin{aligned} u_1(F,F) &> u_1(M,F) \\ u_2(F,F) &> u_2(F,M) \end{aligned}$$

Best response sets:

$$\begin{aligned} B_1(M) &= \{M\} \\ B_1(F) &= \{F\} \\ B_2(M) &= \{M\} \\ B_2(F) &= \{F\} \end{aligned}$$

Since $M \in B_1(M)$ and $M \in B_2(M)$, (M, M) is a Nash equilibrium. Similarly since $F \in B_1(F)$ and $F \in B_2(F)$, (F, F) is a Nash equilibrium.

The profile (M, F) is not a Nash equilibrium since,

$$\begin{aligned} M &\notin B_1(F) \\ F &\notin B_2(M) \end{aligned}$$

Example 2: Prisoner's Dilemma

	2	
1	NC	C
NC	$-2, -2$	$-10, -1$
C	$-1, -10$	$-5, -5$

Note that (C, C) is the unique Nash equilibrium here. To see why, look at the best response sets:

$$\begin{aligned} B_1(C) &= \{C\} \\ B_1(NC) &= \{C\} \\ B_2(C) &= \{C\} \\ B_2(NC) &= \{C\} \end{aligned}$$

Since $s_i^* \in B_1(s_2^*)$ and $s_2^* \in B_2(s_1^*)$ for a Nash equilibrium, the only possible Nash equilibrium here is (C, C). In fact as already seen, this is a strongly dominant strategy equilibrium.

Result: Given a pure strategy game $\Gamma = (N, (S_i), (u_i))$, a strongly dominant strategy equilibrium $(s_1^*, \ldots s_n^*)$ is also a Nash equilibrium.

Proof: Since $(s_1^*, \ldots, s_n^*)$ is a strongly dominant strategy equilibrium, $\forall i = 1, \ldots, n$,

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \quad \forall s_i \in S_i \setminus \{s_i^*\}, \quad \forall s_{-i} \in S_{-i}$$

In particular, choose $s_{-i} = s^*_{-i}$ we have

$$u_i(s_i^*, s^*_{-i}) > u_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i \setminus \{s_i^*\}, \quad \forall s_{-i} \neq s_{-i}*$$

This implies

$$u_i(s_i^*, s^*_{-i}) \geq u_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i$$

Therefore $(s_1^*, \ldots, s_n^*)$ is a Nash equilibrium.
On similar lines, one can prove the following result.

Result: Given a game $\Gamma = (N, (S_i), (u_i))$, a weakly dominant strategy equilibrium $(s_1^*, \ldots, s_n^*)$ is also a Nash equilibrium.

Observation: It is obvious that a Nash equilibrium need not be a weakly dominant or strongly dominant strategy equilibrium. It fact, Nash equilibrium is a much weaker solution concept.

Example 3: Matching Pennies

	2	
1	H	T
H	−1, +1	+1, −1
T	+1, −1	−1, +1

This game does not have a pure strategy Nash equilibrium.

Example 4: Hawk-Dove

	2	
1	H	D
H	3,3	4,1
D	1,4	0,0

Unique Nash equilibrium is given by (H, H).

Example 5: Coordination Game

	2	
1	M	F
M	2,2	0,0
F	0,0	1,1

This game has two Nash equilibria, namely (M, M) and (F, F).

Example 6: Cold War

India	Pak	
	Welfare	Defence
Welfare	10,10	-10, 20
Defence	20, -10	0,0

There is a unique Nash equilibrium namely (Defence, Defence).

Example 7: Tragedy of the the Common

$$N = \{1, 2, \ldots, n\} \text{ is a set of farmers}$$
$$S_1 = S_2 = \cdots = S_n = \{0, 1\}$$

1 corresponds to keeping a sheep, and 0 corresponds to not keeping a sheep. Keeping a sheep gives a benefit of 1. However, when a sheep is kept, damage to the environment is 5. This damage is equally shared by all the farmers.
For $i = 1, 2, \ldots, n$

$$u_i(s_1, \ldots, s_n) = s_i - \frac{5}{n}\sum_{j=1}^{n} s_j = \left(\frac{n-5}{n}\right) s_i - \frac{5}{n}\sum_{j \neq i} s_j$$

<u>Case 1</u>: $n < 5$.
Given any $s_{-i} \in S_{-i}$,

$$u_i(0, s_{-i}) = -\frac{5}{n}\sum_{j \neq i} s_j$$
$$u_i(1, s_{-i}) = \left(\frac{n-5}{n}\right) - \frac{5}{n}\sum_{j \neq i} s_j$$

since $n < 5$, $\left(\frac{n-5}{n}\right) < 0$, and therefore, $u_i(0, s_{-i}) > u_i(1, s_{-i}) \quad \forall s_{-i} \in S_{-i}$. This implies that

$$B_i(s_{-i}) = \{0\} \quad \forall i \in N$$

This means $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium. That is, there is no incentive for any farmer to keep a sheep.

<u>Case 2</u>: $n = 5$.
Here

$$u_i(0, s_{-i}) = -\frac{5}{n}\sum_{j \neq i} s_j$$
$$u_i(1, s_{-i}) = -\frac{5}{n}\sum_{j \neq i} s_j$$

Thus

$$u_i(0, s_{-i}) = u_i(1, s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

This implies

$$B_i(s_{-i}) = \{0, 1\} \quad \forall s_{-i} \in S_{-i}$$

Also it can be seen that all the strategy profiles are Nash Equilibria here. Also note that they are neither weakly dominant nor strongly dominant strategy equilibria.

Case 3: $n > 5$. Here

$$\begin{aligned} u_1(0, s_{-i}) &= -\frac{5}{n} \sum_{j \neq i} s_j \\ u_i(1, s_{-i}) &= \frac{n-5}{n} - \frac{5}{n} \sum_{j \neq i} s_j \end{aligned}$$

Thus

$$u_i(1, s_{-i}) > u_i(0, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

This implies that

$$B_i(s_{-i}) = \{1\} \quad \forall i \in N$$

Hence $(1, 1, \ldots, 1)$ is a strongly dominant strategy equilibrium. Thus if $n > 5$, it is good for all the farmers to keep a sheep.

Now if the Government decides to impose a pollution tax of 5 units for each sheep kept, we have

$$u_i(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5}{n} \sum_{j=1}^{n} s_j = -4s_i - \frac{5}{n} s_i - \frac{5}{n} \sum_{j \neq i} s_j$$

Here

$$\begin{aligned} u_i(0, s_{-i}) &= -\frac{5}{n} \sum_{j \neq i} s_j \\ u_i(1, s_{-i}) &= -4 - \frac{5}{n} - \frac{5}{n} \sum_{j \neq i} s_j \\ \therefore B_i(s_{-i}) &= \{0\} \quad \forall i \in N \end{aligned} \tag{1}$$

This means whatever the value of n, $(0, 0, \ldots, 0)$ is a strongly dominant strategy equilibrium. This is bad news for the farmers.

Interpretations of Nash equilibrium

Note that a Nash equilibrium is a profile of strategies, one for each of the n players, that has the property that each player's choice is his best response to the choices of the other $(n-1)$ players.

- By deviating from a Nash equilibrium strategy, a player is not going to be better off given the Nash equilibrium strategies of the other players.

Several interpretations have been put forward by game theorists. Some of these are given below.

Interpretation 1: Prescription

An adviser or a consultant to the n players would basically advise a Nash equilibrium strategy profile to the players.

- If the adviser prescribes strategies that do not constitute a Nash equilibrium, then some players would find that it would be better for them to do differently than advised.
- If the adviser prescribes strategies that do constitute a Nash equilibrium, then all players are kept happy because there don't have to deviate from the strategies.

Thus a logical, rational, "good" adviser will advise Nash equilibria.

Interpretation 2: Prediction

If the players are rational and intelligent, then a Nash equilibrium is a good prediction for the game.

- For example, iterated elimination of strongly dominated strategies will lead to a reduced form which will include a Nash equilibrium.
- In the Cournot's pricing game, iterated elimination of strongly dominated strategies leads to a unique prediction, the Nash equilibrium.

Interpretation 3: Self Enforcing Agreement

A Nash equilibrium can be viewed as an implicit or explicit agreement between the players. Once this agreement is reached, it does not need any external means of enforcement because it is in the self-interest of each player to follow this agreement if the others do.

- In a non-cooperative game, since agreements cannot be enforced, Nash equilibrium agreements are the only ones sustainable.

Interpretation 4: Evolution and steady-state

A Nash equilibrium is a potential stable point of a dynamic adjustment process in which players adjust their behavior to that of other players in the game, constantly searching for strategy choices that will give them the best results.

- This argument is used to explain biological evolution.
- In this interpretation, Nash equilibrium is the outcome that results over time when a game is played repeatedly.
- Nash equilibrium is like a stable social convention that people are happy to maintain forever.

Focal Point Effect

- If a game has multiple Nash equilibria, then which of these would be implemented by the players?
- This question was investigated by Schelling (1960) who proposed the focal point effect
  - Schelling's argument is that anything that tends to focus the player's attention on one equilibrium may make them all expect it and hence fulfill it, like a self-fulfilling prophecy.
  - Such a Nash equilibrium, which has some property that conspicuously distinguishes it from all other equilibria is called a focal equilibrium.

Example: BOS game

	2	
1	M	F
M	2,1	0,0
F	0,0	1,2

Here (M, M) and (F, F) are both Nash equilibria. If an extremely popular music concert is announced, then (M, M) may become the focal equilibrium. On the other hand, if an extremely popular film is running in town and everybody is talking about it, then (F, F) may become the focal equilibrium.