Game Theory

Lecture Notes By

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COOPERATIVE GAME THEORY
3. Coalitional Games: Introduction

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

In the previous chapter, we have studied the two person bargaining problem where we have explored the effect of cooperation between two players. In this chapter, we introduce multiplayer coalitional games.

Multi-Person Bargaining Problem

Let $N = \{1, \ldots, n\}$ be the set of players. We have already seen the Nash bargaining solution for a two player game. What will this solution look like for an n-player game with $n > 2$? Let F be the set of feasible allocations that the players can get if they all work together. Let us assume that F is a closed convex subset of $\Re^n$. Let $(v_1, \ldots, v_n)$ be the disagreement payoff allocation the players would expect if they did not cooperate. Also assume that the set $\{(y_1, \ldots, y_n) \in F : y_i \geq v_i \; \forall i \in N\}$ is non-empty and bounded. The pair $(F, (v_1, \ldots, v_n))$ is then called an *n-person bargaining problem*. The bargaining problem $(F, (v_1, \ldots, v_n))$ is said to be *essential* if there exists $y \in F$ such that $y_i > v_i \; \forall i \in N$.

Suppose $(F, (v_1, \ldots, v_n))$ is essential. Then its Nash bargaining solution can be defined to be the unique strongly efficient allocation vector that maximizes

$$\prod_{i \in N}^{n} (x_i - v_i)$$

over all vectors $x \in F$ such that $x_i \geq v_i \; \forall i \in N$.

However this Nash bargaining solution ignores the possibility of cooperation among subsets of the players as shown in the series of three examples below. Consequently for $n > 2$, Nash bargaining solution may not give a good solution. This motivates the study of coalitions in cooperative games with $n > 2$.

Example 1: Divide the Dollar Game - Version 1

Here there are three players, so $N = \{1, 2, 3\}$. The players wish to divide 300 units of money among themselves. Each player can propose a payoff such that no player's payoff is negative and the sum of all the payoffs does not exceed 300. The strategy sets can therefore be defined as follows:

$$S_1 = S_2 = S_3 = \{(x_1, x_2, x_3) \in \Re^3 : x_1 + x_2 + x_3 \leq 300; \;\; x_1 \geq 0; \; x_2 \geq 0, \; x_3 \geq 0\}$$

Assume that the players will get 0 unless all three players propose the same allocation. That is for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) \; &= x_i \quad \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Note in this game that the players can achieve any allocation in which their payoffs are non-negative and sum to $\leq$ 300 and the minimax value guaranteed for each player is 0. The above game can therefore be described as a three person bargaining problem (F, v) where:

$$\begin{aligned} F \; &= \; \{(x_1, x_2, x_3) \in \Re^3 : x_1 + x_2 + x_3 \leq 300, x_1 \geq 0; \; x_2 \geq 0, \; x_3 \geq 0\} \\ v \; &= \; (v_1, v_2, v_3) = (0, 0, 0) \end{aligned}$$

The Nash bargaining solution for this problem is $x = (100, 100, 100)$, which is a reasonable outcome for this situation.

Example 2: Divide the Dollar Game - Version 2

This is a slight variation of Version 1 with the difference that players get 0 unless player 1 and player 2 propose the same allocation in which case they get this allocation. That is, for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) \; &= x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The same bargaining problem (F, v) as in Example 1 would describe the situation here and hence the Nash bargaining solution for this problem also is $x = (100, 100, 100)$.

- This solution looks quite unreasonable at first look. This is because players 1 and 2 together determine the payoff allocation and player 3 is not involved in this. So, we would expect players 1 and 2 to simply divide the payoff equally between them, leading to the allocation $(150, 150, 0)$. Another viewpoint which reinforces this is as follows. Suppose 1 and 2 ignore 3 and played out a two person cooperative game. Such a game would have the Nash bargaining solution that divides 300 equally between 1 and 2.

- What could be some factors that might suggest $(100, 100, 100)$ as a solution for this problem ?

  1. The players must choose their proposals simultaneously and both $(100, 100, 100)$ and $(150, 150, 0)$ are equilibria for the players.
  2. Even if there is costless, non-binding preplay communication between the players, there is still an equilibrium in which both players 1 and 2 expect expect each other to ignore anything that was said. This again points to $(100, 100, 100)$ as a possibility.

3. If player 3 has any influence, then he would certainly influence players 1 and 2 to go for the equilibrium $(100, 100, 100)$.

- However, there is one key assumption based on which the outcome $(100, 100, 100)$ can be dropped. This is the *effective negotiation* assumption. Intuitively this means that players 1 and 2 can negotiate in an effective way and focus themselves on an equilibrium that they prefer. In particular, they gravitate towards the $(150, 150, 0)$ equilibrium.

- The members of a coalition of players are said to *negotiate effectively* if: the players, on realizing that there is a feasible change in their strategies that would benefit them all, they all would agree to actually make such a change unless such a change contradicts some agreements that some members of the coalition might have made with other players outside this coalition, in the context of some other equally effective coalition.

- *Effective Negotiation is the key assumption that distinguishes cooperative game theory from non-cooperative game theory.*

- The n-person Nash bargaining solution would be relevant if the only coalition that can negotiate effectively is the *grand coalition* that includes the whole of N. If other coalitions also can negotiate effectively, then the Nash solution is no longer as good. This is because it suppresses all information about the power of multi-player coalitions other than the grand coalition N.

- In Example 1, no coalition that is smaller than $\{1, 2, 3\}$ can guarantee more than 0 to its members. In Example 2, the coalition $\{1, 2\}$ could guarantee its members any payoff allocation that they could get in $\{1, 2, 3\}$. In general, cooperative game theory becomes complex because of possibility of competition between overlapping coalitions.

Example 3: Divide the Dollar Game - Version 3

This is a slight variation of Version 2 with the difference that players get 0 unless player 1 and player 2 propose the same allocation or player 1 and player 3 propose the same allocation, in which case they get what is proposed. That is, for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3) or s_1 = s_3 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The bargaining problem (F, v) as in Examples 1 and 2 would describe the situation here and hence the Nash bargaining solution for this problem also is $x = (100, 100, 100)$. Like in the case of Example 2, this solution looks quite unreasonable since players 1 and 2 together or players 1 and 3 together determine the payoff allocation. Player 1 is necessarily involved in both the above situations. So, we would expect the players to divide the payoff in a way that players 2 and 3 get the same payoff but this payoff is less than the payoff that player 1 would get (since player 1 has to necessarily agree for a non-zero allocation). This leads to uncountably infinite number of possibilities, such as (120, 90, 90), (150, 75, 75), (200, 50, 50), (280, 10, 10), etc. One can even suggest an allocation (300, 0, 0) on the ground that player 1 is indispensable for a non-zero allocation.

Example 4: Divide the Dollar Game - Version 3 (Majority Voting Game)

The difference with the previous versions is that the players get 0 unless there is some pair of players $\{1,2\},\{2,3\}$, or $\{1,3\}$ who propose the same allocation, in which case they get this allocation. That is,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_j = s_k = (x_1, x_2, x_3) \text{ for some } j \neq k \\ &= 0 \quad \text{otherwise} \end{aligned}$$

- Here again, the Nash bargaining solution is $(100, 100, 100)$. This looks quite reasonable for this version because of the symmetry and the equal bargaining power of the players. Observe that this allocation is a Nash equilibrium.
- If we assume that every coalition can negotiate effectively, the analysis becomes quite interesting as seen below.
- If players 1 and 2 negotiate effectively in the coalition $\{1,2\}$, they can agree to the allocation $(150, 150, 0)$ which is better for both of them. Observe that this allocation is also a Nash equilibrium.
- If $(150, 150, 0)$ is the expected outcome, then player 3 would be eager to persuade player 1 or player 2 to form an effective coalition with him. For example, player 3 would be willing to negotiate an agreement with player 2 to both propose $(0, 225, 75)$. This allocation is also a Nash equilibrium.
- If $(0, 225, 75)$ were to be the expected outcome in the absence of further negotiations, then player 1 would be willing to negotiate an agreement with player 3 to propose an allocation that is better for both of them, say, $(113, 0, 187)$. This allocation is again a Nash equilibrium.
- It turns out that in any equilibrium of this game, there is always at least one pair of players who would both do strictly better by jointly agreeing to change their strategies together.

The above sequence of coalitional negotiations seems to have no end. There are two possible ways in which the negotiations could conclude.

1. Let us say that a player, having negotiated an agreement as part of a coalition, cannot later negotiate a different agreement with another coalition, that does not contain all the members of the first coalition. For example, if the grand coalition $\{1,2,3\}$ negotiated the agreement $(100, 100, 100)$ before any two player coalition could negotiate separately, then no two player coalition can undercut this outcome. Also, if players 1 and 2 first negotiated an agreement $(150, 150, 0)$, then player 3 would be unable to increase this payoff by negotiating with player 1 or player 2 separately. It is clear that the order in which coalitions can meet and negotiate will crucially determine the outcome of the game. The advantage lies with coalitions that negotiate earlier.
2. Suppose the negotiated agreements are tentative and non-binding. Thus a player who negotiates in a sequential manner in various coalitions can nullify his earlier agreements and reach a different agreement with a coalition that negotiates later. Here the order in which negotiations are made and nullified will have a bearing on the final outcome. For example, let us say the order of negotiations is $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ and $\{1,2,3\}$. Here any agreement by $\{1,2\}$ and $\{2,3\}$ in

that order to pay non-zero amount to player 2 can be overturned by the coalition $\{1, 3\}$ which might agree on $(150, 0, 150)$. Player 2 may not be able to make them concede anything when the turn of coalition $\{1, 2, 3\}$ arises.

3. As another example, assume that player 1 believes that, any negotiated agreement with player 2, would be overturned by player 3. Player 1 may first suggest $(100, 100, 100)$ and stick to it and refuse to agree for $(150, 150, 0)$. This he would do to prevent any possibility of his getting zero payoff. It is clear that coalitions that get to negotiate later hold the advantage in this scheme.

4. In realistic cooperative scenarios where different coalitions could form, the number of possibilities could be mind-boggling and a systematic analysis of all scenarios may be prohibitive. There are as many as $2^n - 1$ such coalitions possible and therefore there is a need for theories of cooperative games that can provide some sense of what to expect as a result of the balance of power among various coalitions. Such order-independent theories will be extremely useful but will pose challenges in interpretation because ordering is often natural and important.

Games in Characteristic Form (TU Games)

The assumption of *transferable utility* makes cooperative games somewhat tractable. This assumption implies that there is a commodity called money that the players can freely transfer among themselves such that any player's payoff increases by one unit for every unit of money that he gets. With the assumption of transferable utility in place, the cooperative possibilities of a game can be described by a *characteristic function* $v : 2^N \rightarrow \Re$, that assigns a number $v(C)$ to every coalition $C \subseteq N$. $v(\emptyset)$ is always taken to be zero. $v(C)$ is called the *worth* of the coalition C and it captures the total amount of transferable utility that the members of C could earn without any help from the players outside of C.

Definition: A cooperative game with transferable utility is defined as the pair (N, v) where $N = \{1, \ldots, n\}$ is a set of players and $v : 2^N \rightarrow \Re$ is a characteristic function, with $v(\emptyset) = 0$. We call such a game also as a *game in coalition form*, *game in characteristic form*, or *coalitional game* or *TU game*.

Note that, under the assumption of transferable utility, specifying a single number for each coalition is enough to describe what allocations of utility can be achieved by the members of the coalition.

Non-Transferable Utility (NTU) Games

In contrast, games without transferable utility (also called NTU coalitional games or games in NTU coalitional form) are defined as follows.

Definition: An NTU coalitional game on the set of players N is any mapping $V(.)$ on the domain $L(N)$ such that, for any coalition $C \subset N$,

- $V(C)$ is a non-empty closed and convex set of $\Re^{|C|}$, and
- The set $\{x : x \in V(C) \text{ and } x_i \geq v_i \; \forall i \in C\}$ is a bounded subset of $\Re^{|C|}$, where

$$\begin{aligned} v_i &= \max\{y_i : y \in V(\{i\})\} \\ &< \infty \;\; \forall i \in N \end{aligned}$$

Here $V(C)$ is the set of expected payoff allocations that the members of coalition C could guarantee for themselves if they act cooperatively. An NTU game is a generalization of a TU game.

In the remainder of the discussion, we will consider only TU games.

Example 5: Characteristic Functions for Divide the Dollar Games

The Divide-the-Dollar - Version 1 game discussed in Example 1 has the following characteristic function.

$$\begin{aligned} v(\{1,2,3\}) &= 300 \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

Version 2 of the game has the characteristic function:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = 300 \\ v(\{2,3\}) &= v(\{1,3\}) = 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

Version 3 of the game has the characteristic function:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = v(\{1,3\}) = 300 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0 \end{aligned}$$

Version 4 of the game (majority voting game) has the characteristic form:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 300 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

Representations for TU Games

Like in the case of Nash bargaining problems, there are three different ways in which the characteristic function could be defined for TU games, starting from the base model, namely a strategic form game.

Minimax Representation

Let $\Gamma = \langle N, (S_i), (u_i) \rangle$ be a n-person strategic form game with transferable utility. Let $C \subset N$ be any coalition of players. $N \setminus C$ is the set of all players who are not in the coalition C. Let

$$S_{N \setminus C} = \times_{j \in N \setminus C} \; S_j$$

$$S_C = \times_{j \in C} \; S_j$$

Now, $\Delta(S_C)$ is the set of correlated strategies available to coalition C. Let $u_i(\sigma_C, \sigma_{N \setminus C})$ denote player i's expected payoff, before transfers of money, when the correlated strategies σ_C and $\sigma_{N \setminus C}$ are independently implemented:

$$u_i(\sigma_C, \sigma_{N \setminus C}) = \sum_{s_C \in S_C} \; \sum_{s_{N \setminus C} \in S_{N \setminus C}} \sigma_C(s_C) \sigma_{N \setminus C}(s_{N \setminus C}) \; u_i(s_C, s_{N \setminus C})$$

It was suggested by von Neumann and Morgenstern that the characteristic function should be defined by:

$$v(C) = \min_{\sigma_{N\setminus C} \in \Delta(S_{N\setminus C})} \max_{\sigma_C \in \Delta(S_C)} \sum_{i \in C} u_i(\sigma_C, \sigma_{N\setminus C})$$

$v(C)$ may be interpreted as the maximum sum of utility payoffs that the members of coalition C can guarantee themselves against the best offensive threat by the complementary coalition. This is called the minimax representation in coalitional form of the strategic form game Γ with transferable utility.

Intuition on Minimax Representation

- This representation implicitly assumes that a coalition C should be concerned that $N \setminus C$ would attack C offensively if the members of C decided to cooperate with each other but without the players in $N \setminus C$.
- However, offensively minimizing the sum of payoffs of the players in C may not be in the best interests of players in $N \setminus C$. Note that the primary interest of all players is to maximize their own payoffs.
- A justification for the assumption that the members of $N \setminus C$ might act offensively against C is as follows. When all players ultimately cooperate together as a part of the grand coalition N and the players are negotiating over the possible division of worth $v(N)$, the players in $N \setminus C$ can jointly commit themselves to an offensive threat that would be carried out only in the improbable event that the players in C break off negotiations with players in $N \setminus C$.
- Such a threat by $N \setminus C$ is a deterrent on the coalition C and the members of C would be willing to concede a larger share to $N \setminus C$.

Defensive Equilibrium Representation

Here we assume that complementary coalitions would play an essentially defensive pair of equilibrium strategies against each other. The implicit assumption that each coalition makes here is that the complementary coalition will play an equilibrium strategy and the coalition settles for a defensive strategy by playing *its* equilibrium strategy. $C \subset N$. Then $N \setminus C$ is the complementary coalition. For all $C \subset N$, define $\bar{\sigma}_C$ as a correlated strategy belonging to the set

$$\overset{\text{argmax}}{\sigma_C \in \Delta(S_C)} \sum_{i \in C} u_i(\sigma_C, \bar{\sigma}_{N\setminus C})$$

Similarly, define $\bar{\sigma}_{N\setminus C}$ a correlated strategy belonging to the set

$$\overset{\text{argmax}}{\sigma_{N\setminus C} \in \Delta(S_{N\setminus C})} \sum_{j \in N\setminus C} u_j(\bar{\sigma_C}, \sigma_{N\setminus C})$$

Define the characteristic function as

$$v(C) = \sum_{i \in C} u_i(\bar{\sigma_C}, \bar{\sigma}_{N\setminus C})$$

$$v(N \setminus C) = \sum_{j \in N\setminus C} u_j(\bar{\sigma_C}, \bar{\sigma}_{N\setminus C})$$

Then v is called a defensive equilibrium representation in coalition form of the strategic form game Γ with transferable utility.

Rational Threats Representation

This was proposed by Harsanyi in 1963. This representation is derived by generalizing the rational threats criterion of Nash. Let $C \subset N$ be any coalition. Define $\bar{\sigma}_C$ as a correlated strategy belonging to the set

$$\overset{\text{argmax}}{\sigma_C \in \Delta(S_C)} \left[\sum_{i \in C} u_i(\sigma_C, \bar{\sigma}_{N\setminus C}) - \sum_{j \in N\setminus C} u_j(\sigma_C, \bar{\sigma}_{N\setminus C}) \right]$$

Similarly, define $\bar{\sigma}_{N\setminus C}$ a correlated strategy belonging to the set

$$\overset{argmax}{\sigma_{N\setminus C} \in \Delta(S_{N\setminus C})} \left[\sum_{j \in N\setminus C} u_j(\bar{\sigma_C}, \sigma_{N\setminus C}) - \sum_{i \in C} u_i(\bar{\sigma}_C, \sigma_{N\setminus C}) \right]$$

Now define

$$v(C) = \sum_{i \in C} u_i(\bar{\sigma_C}, \bar{\sigma}_{N\setminus C})$$

$$v(N \setminus C) = \sum_{j \in N\setminus C} u_j(\bar{\sigma_C}, \bar{\sigma}_{N\setminus C})$$

Some Observations

Note that in all the three representations, the worth of the grand coalition N is the same:

$$v(N) = \max_{s_N \in S_N} \sum_{i \in N} u_i(s_N)$$

The distinction between the three representations can be interpreted in terms of different assumptions about the ability of the coalitions to commit themselves to offensive and defensive threats.

Example: Different Representations of TU Games

Let $N = \{1, 2, 3\}$; $S_i = \{a_i, b_i\}$ for $i = 1, 2, 3$. Suppose the payoff matrix is as shown below.

	$S_2 \times S_3$			
S_1	a_2, a_3	b_2, a_3	a_2, b_3	b_2, b_3
a_1	4, 4, 4	2, 5, 2	2, 2, 5	0, 3, 3
b_1	5, 2, 2	3, 3, 0	3, 0, 3	1, 1, 1

a_i is to be interpreted as a *generous strategy* and b_i as a *selfish strategy*.

1. In the minimax representation, each coalition C gets the most that it could guarantee itself if the players in the complementary coalition were selfish.

$$\begin{aligned} v(\{1,2,3\}) &= 12 \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 1 \end{aligned}$$

2. In the defensive equilibrium representation, the members of a two player coalition can actually increase the sum of their payoffs by both being generous. Here the characteristic function would be

$$\begin{aligned} v(\{1,2,3\}) &= 12 \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 5 \end{aligned}$$

   This representation imputes an advantage to a player who acts selfishly alone against a generous two player coalition.

3. In the rational threats representation, both the offensive and defensive considerations are taken into account. Here all coalitions smaller than N choose selfishness in the game.

$$\begin{aligned} v(\{1,2,3\}) &= 12 \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 2 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 1 \end{aligned}$$

If all three representations coincide, the game is said to have *orthogonal coalitions*. As to which representation is to be used depends on what the modeler or designer perceives as most appropriate. In some cases, as in the orthogonal case, there may be a natural representation that might immediately suggest itself.

Super-Additivity

A characteristic function v is said to be *super-additive* if, for $\forall C, D \subset N$

$$C \cap D = \emptyset \Longrightarrow v(C \cup D) \geq v(C) + v(D)$$

If v is the minimax representation, then v can be shown to be super-additive. The defensive equilibrium and rational threats representations are not necessarily super-additive.

Given a game (N, v) in coalitional form, the *super-additive cover* of v is the super-additive game (N, w) in coalitional form with the lowest possible worths for all coalitions such that

$$\omega(C) \geq v(C) \ \ \forall C \subset N$$

Let $P(C)$ be the set of all partitions of C. The super-additive cover ω of the game v in coalitional form satisfies the equation

$$\omega(C) = \max \left\{ \sum_{j=1}^{k} v(T_j) : \{T_1, \ldots, T_k\} \in P(C) \right\} \quad \forall C \subset N$$

The above definition implies that the worth of a coalition in the super-additive cover is the maximum worth that the coalition could achieve by breaking up into a set of smaller disjoint coalitions. The notion of a super-additive cover provides a way to define a super-additive game corresponding to any game in coalitional form.

To Probe Further

The material discussed in this chapter draws upon mainly from the the book by Myerson [1].

References

[1] Roger B. Myerson. *Game Theory: Analysis of Conflict*. Harvard University Press, 1997.