Chapter 5: Pure Strategy Nash Equilibrium

Note: This is a only a draft version, so there could be flaws. If you find any errors, please do send email to hari@csa.iisc.ernet.in. A more thorough version would be available soon in this space.

Nash Equilibrium

John Nash (who won the Nobel Prize in 1994) is credited with the invention of Nash equilibrium, one of the most celebrated and brilliant notions in game theory [1, 2].

Dominant strategy equilibria (strongly dominant, weakly dominant), if they exist, are very desirable but rarely do they exist because the conditions to be satisfied are too demanding. In a two player game, a dominant strategy equilibrium requires that each player’s choice be optimal against all choices of the other player. If we only insist that is optimal for the optimal choices of the other player, we get a Nash equilibrium.

Definition

Given a game $\Gamma = \langle N, (S_i), (u_i) \rangle$ with pure strategies, the strategy profile

$$s^* = (s_1^*, s_2^*, \ldots, s_n^*)$$

is said to be a pure strategy Nash equilibrium of $\Gamma$ if,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i, \quad \forall i = 1, 2, \ldots, n.$$ 

That is, each player’s Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players.

Recall the notation

$$\Gamma = \langle N, (S_i), (u_i) \rangle$$

$$N = \{1, 2, \ldots, n\}$$

$$S = S_1 \times S_2 \times \cdots \times S_n$$

$$s = (s_1, s_2, \ldots, s_n) \in S$$
\[ S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \]
\[ s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \]
\[ u_i : S_1 \times \cdots \times S_n \rightarrow \mathbb{R} \]

**Best Response Correspondence for player** \( i \): Given a game \( \Gamma = \langle N, (S_i), (u_i) \rangle \), the best response correspondence for player \( i \) is the mapping

\[ B_i : S_{-i} \rightarrow 2^{S_i} \]

defined by

\[ B_i(s_{-i}) = \{ s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}), \quad \forall s'_i \in S_i \} \]

That is, given a profile \( s_{-i} \) of strategies of the other players, \( B_i(s_{-i}) \) gives the set of all best response strategies of player \( i \).

**Alternative Definition of Nash Equilibrium**

Given a pure strategy game \( \Gamma = \langle N, (S_i), (u_i) \rangle \), the strategy profile \( (s^*_1, \ldots, s^*_n) \) is a Nash equilibrium iff,

\[ s^*_i \in B_i(s^*_{-i}), \quad \forall i = 1, \ldots, n \]

**Note:** For \( (s^*_1, \ldots, s^*_n) \) to be a Nash equilibrium, it must be that no player can profitably deviate from his Nash equilibrium strategy given that the other players are playing their Nash equilibrium strategies.

**Example 1: The BOS Game**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>M</td>
<td>F</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
<td>0, 0</td>
</tr>
<tr>
<td>F</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

There are two Nash equilibria here, namely (M,M) and (F,F). The profile (M,M) is Nash equilibrium because

\[ u_1(M, M) > u_1(F, M) \]
\[ u_2(M, M) > u_2(M, F) \]

The profile (F,F) is a Nash equilibrium because

\[ u_1(F, F) > u_1(M, F) \]
\[ u_2(F, F) > u_2(F, M) \]
Best response sets:

\[ B_1(M) = \{M\} \]
\[ B_1(F) = \{F\} \]
\[ B_2(M) = \{M\} \]
\[ B_2(F) = \{F\} \]

Since \( M \in B_1(M) \) and \( M \in B_2(M) \), \((M, M)\) is a Nash equilibrium. Similarly since \( F \in B_1(F) \) and \( F \in B_2(F) \), \((F, F)\) is a Nash equilibrium.

The profile \((M, F)\) is not a Nash equilibrium since,

\[ M \notin B_1(F) \]
\[ F \notin B_2(M) \]

Example 2: Prisoner’s Dilemma

\[
\begin{array}{c|cc}
& 1 & 2 \\
\hline
NC & -2, -2 & -10, -1 \\
C & -1, -10 & -5, -5 \\
\end{array}
\]

Note that \((C, C)\) is the unique Nash equilibrium here. To see why, look at the best response sets:

\[ B_1(C) = \{C\} \]
\[ B_1(NC) = \{C\} \]
\[ B_2(C) = \{C\} \]
\[ B_2(NC) = \{C\} \]

Since \( s_i^* \in B_1(s_j^*) \) and \( s_j^* \in B_2(s_i^*) \) for a Nash equilibrium, the only possible Nash equilibrium here is \((C, C)\). In fact as already seen, this is a strongly dominant strategy equilibrium.

Result: Given a pure strategy game \( \Gamma = \langle N, (S_i, (u_i)) \rangle \), a strongly dominant strategy equilibrium \((s_1^*, \ldots, s_n^*)\) is also a Nash equilibrium.

Proof: Since \((s_1^*, \ldots, s_n^*)\) is a strongly dominant strategy equilibrium, \( \forall i = 1, \ldots, n, \)

\[ u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}), \quad \forall s_i \in S_i \setminus \{s_i^*\}, \quad \forall s_{-i} \in S_{-i} \]

In particular, choosing \( s_{-i} = s_{-i}^* \), we have

\[ u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i \setminus \{s_i^*\}, \quad \forall s_{-i} \neq s_{-i}^* \]

This implies

\[ u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \quad \forall s_i \in S_i \]
Therefore \((s_1^*,\ldots,s_n^*)\) is a Nash equilibrium. On similar lines, one can prove the following result.

**Result:** Given a game \(\Gamma = (N,(S_i),(u_i))\), a weakly dominant strategy equilibrium \((s_1^*,\ldots,s_n^*)\) is also a Nash equilibrium.

**Observation:** It is obvious that a Nash equilibrium need not be a weakly dominant or strongly dominant strategy equilibrium. It fact, Nash equilibrium is a much weaker solution concept.

**Example 3: Matching Pennies**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>T</td>
<td>+1</td>
<td>-1</td>
</tr>
</tbody>
</table>

This game does not have a pure strategy Nash equilibrium.

**Example 4: Hawk-Dove**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Unique pure strategy Nash equilibrium is given by \((H,H)\).

**Example 5: Coordination Game**

<table>
<thead>
<tr>
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<th>1</th>
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</thead>
<tbody>
<tr>
<td>M</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

This game has two Nash equilibria, namely \((M,M)\) and \((F,F)\).

**Example 6: Cold War**

<table>
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<tr>
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<th>Pak</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Welfare</td>
<td>Defence</td>
</tr>
<tr>
<td>India</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Welfare</td>
<td>10, 10</td>
<td>-10, 20</td>
</tr>
<tr>
<td>Defence</td>
<td>20, -10</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
There is a unique Nash equilibrium namely (Defence, Defence).

Example 7: Tragedy of the Commons

Recall that

\[ N = \{1, 2, \ldots, n\} \] is a set of farmers
\[ S_1 = S_2 = \cdots = S_n = \{0, 1\} \]

1 corresponds to keeping a sheep, and 0 corresponds to not keeping a sheep. Keeping a sheep gives a benefit of 1. However, when a sheep is kept, damage to the environment is 5. This damage is equally shared by all the farmers.

For \( i = 1, 2, \ldots, n \)

\[ u_i(s_1, \ldots, s_n) = s_i - \frac{5}{n} \sum_{j=1}^{n} s_j = \left(\frac{n-5}{n}\right)s_i - \frac{5}{n} \sum_{j \neq i} s_j \]

Case 1: \( n < 5 \).

Given any \( s_{-i} \in S_{-i} \),

\[ u_i(0, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j \]
\[ u_i(1, s_{-i}) = \left(\frac{n-5}{n}\right) - \frac{5}{n} \sum_{j \neq i} s_j \]

since \( n < 5, \; \left(\frac{n-5}{n}\right) < 0 \), and therefore, \( u_i(0, s_{-i}) > u_i(1, s_{-i}) \) \( \forall s_{-i} \in S_{-i} \). This implies that

\[ B_i(s_{-i}) = \{0\} \; \forall i \in N \]

This means \( (0, 0, \ldots, 0) \) is a strongly dominant strategy equilibrium. That is, there is no incentive for any farmer to keep a sheep.

Case 2: \( n = 5 \).

Here

\[ u_i(0, s_{-i}) = \frac{5}{n} \sum_{j \neq i} s_j \]
\[ u_i(1, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j \]

Thus

\[ u_i(0, s_{-i}) = u_i(1, s_{-i}), \; \forall s_{-i} \in S_{-i} \]

This implies

\[ B_i(s_{-i}) = \{0, 1\} \; \forall s_{-i} \in S_{-i} \]

Also it can be seen that all the strategy profiles are Nash Equilibria here. Also note that they are neither weakly dominant nor strongly dominant strategy equilibria.
Case 3: $n > 5$. Here

\[ u_1(0, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j \]
\[ u_i(1, s_{-i}) = \frac{n - 5}{n} - \frac{5}{n} \sum_{j \neq i} s_j \]

Thus

\[ u_i(1, s_{-i}) > u_i(0, s_{-i}) \quad \forall s_{-i} \in S_{-i} \]

This implies that

\[ B_i(s_{-i}) = \{1\} \quad \forall i \in N \]

Hence $(1,1,\ldots,1)$ is a strongly dominant strategy equilibrium. Thus if $n > 5$, it is good for all the farmers to keep a sheep.

Now if the Government decides to impose a pollution tax of 5 units for each sheep kept, we have

\[ u_i(s_1, \ldots, s_n) = s_i - 5s_i - \frac{5}{n} \sum_{j=1}^{n} s_j = -4s_i - \frac{5}{n} s_i - \frac{5}{n} \sum_{j \neq i} s_j \]

Here

\[ u_i(0, s_{-i}) = -\frac{5}{n} \sum_{j \neq i} s_j \]
\[ u_i(1, s_{-i}) = -4 - \frac{5}{n} - \frac{5}{n} \sum_{j \neq i} s_j \]
\[ \therefore B_i(s_{-i}) = \{0\} \quad \forall i \in N \]

(1)

This means whatever the value of $n$, $(0,0,\ldots,0)$ is a strongly dominant strategy equilibrium. This is bad news for the farmers. 

Example 8: Bandwidth Sharing Problem

This problem is based on an example presented by Tardos and Vazirani [3]. There is a shared communication channel of maximum capacity 1. Each player may wish to send $f_i$ units of flow, where $f_i \in [0,1]$. We have

\[ N = \{1,2,\ldots,n\} \]
\[ S_1 = S_2 = \ldots = S_n = [0,1] \]

This is an infinite game since the strategy sets are real intervals.

If $\sum_{i \in N} f_i > 1$, then the transmission cannot happen since the capacity is exceeded and the payoff to each player is zero. If $\sum_{i \in N} f_i < 1$, then assuming that the following is the payoff to the player $i$:

\[ u_i = f_i(1 - \sum_{j \in N} f_j) \]
The above expression models the fact that the payoff to a player is proportional to the flow sent by the player but is inversely proportional to the total flow. The second term captures the fact that the quality of transmission deteriorates with the total bandwidth used.

The above defines an $n$-player infinite game. We compute a Nash equilibrium for this in the following way. Let $\sum_{j \in N} x_j < 1$ and consider player $i$. Let

$$t = \sum_{j \neq i} x_j$$

The payoff for the player $i$

$$= x_i(1 - t - x_i)$$

In order to maximize the above payoff, we have to choose

$$x_i^* = \arg \max_{x_i \in [0,1]} x_i(1 - t - x_i)$$

$$= \frac{1 - t}{2}$$

$$= \frac{1 - \sum_{j \neq i} x_j^*}{2}$$

If this has to be satisfied for all $i \in N$, then we end up with $n$ simultaneous equations

$$x_i^* = \frac{1 - \sum_{j \neq i} x_j^*}{2} \quad i = 1, 2, \ldots, n$$

It can be shown that the above set of simultaneous equations has the unique solution:

$$x_i^* = \frac{1}{1 + n} \quad i = 1, 2, \ldots, n$$

The profile $(x_1^*, \ldots, x_n^*)$ is thus a Nash equilibrium. However, as shown below, this is not a very happy situation. The payoff for player $i$ in the above Nash equilibrium

$$= \left( \frac{1}{n+1} \right) \left( \frac{1}{n+1} \right)$$

Therefore the total payoff to all players combined

$$= \frac{n}{(n+1)^2}$$

Now, consider the profile

$$\left( \frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n} \right)$$

This profile gives each player a payoff

$$= \frac{1}{2n} \left( 1 - \frac{n}{2n} \right)$$

$$= \frac{1}{4n}$$

Therefore the total payoff to all the players

$$= \frac{1}{4} > \frac{n}{(n+1)^2}$$

Thus a non-equilibrium payoff $(\frac{1}{2n}, \frac{1}{2n}, \ldots, \frac{1}{2n})$ provides more payoff then a Nash equilibrium payoff. This is referred to as a tragedy of the commons.
Example 9: Pricing Game

This example is taken from the article by Tardos and Vazirani[3]. There are two sellers 1 and 2 and there are three buyers A, B, and C.

- A can only buy from seller 1.
- C can only buy from seller 2.
- B can buy from either seller 1 or seller 2.
- Each buyer has a budget (maximum willingness to pay) of 1 and wishes to buy one item.
- The sellers have enough items to sell.
- Each seller announces a price in the range [0, 1].

Let $s_1$ and $s_2$ be the prices announced. Buyer A will buy an item from seller 1 at price $s_1$. Buyer C will buy an item from seller 2 at price $s_2$. If $s_1 \leq s_2$, then buyer B will buy an item from seller 1, otherwise buyer B will buy from seller 2. The game can now be defined as follows.

\[
N = \{1, 2\}
\]

\[
S_1 = S_2 = [0, 1]
\]

\[
u_1(s_1, s_2) = \begin{cases} 
2s_1 & \text{if } s_1 \leq s_2 \\
 s_1 & \text{if } s_1 > s_2
\end{cases}
\]

\[
u_2(s_1, s_2) = \begin{cases} 
2s_2 & \text{if } s_1 > s_2 \\
 s_2 & \text{if } s_1 \leq s_2
\end{cases}
\]

This is a 2-player infinite game. Let us first explore a pure strategy Nash equilibrium for this game. First, we note that $(1, s_2)$ cannot be a Nash equilibrium for any $s_2 \in [1, 0]$. This is because

\[
u_1(1, s_2) = \begin{cases} 
2 & \text{for } s_2 = 1 \\
 1 & \text{for } s_2 < 1
\end{cases}
\]

\[
u_2(1, s_2) = \begin{cases} 
1 & \text{for } s_2 = 1 \\
2s_2 & \text{for } s_2 < 1
\end{cases}
\]

Any $s'_2$ such that $\frac{1}{2} < s_2 < s'_2 < 1$ will ensure that

\[
u_2(1, s'_2) = 2s'_2 > 1 > 2s_2
\]

\[
\therefore u_2(1, s'_2) > u_2(1, s_2)
\]

Similarly, the profile $(s_1, 1)$ cannot be a Nash equilibrium for any $s_1 \in [0, 1]$.

Let $(s^*_1, s^*_2)$ be a pure strategy Nash equilibrium. Note that $s^*_1 \neq 1$ and $s^*_2 \neq 1$. We consider two cases.
Case 1: $s_1^* > \frac{1}{2}$

There are two cases here: (1) $s_1^* \leq s_2^*$ (2) $s_1^* > s_2^*$. Suppose $s_1^* \leq s_2^*$. Then

$$u_1(s_1^*, s_2^*) = 2s_1^*$$
$$u_2(s_1^*, s_2^*) = s_2^*$$

Choose $s_2 \ni \frac{1}{2} < s_2 < s_1^*$. Then

$$u_2(s_1^*, s_2) = s_2^*$$
$$> s_2^* \text{ since } 2s_2 > 1 \text{ and } s_2^* \leq 1$$
$$= u_2(s_1^*, s_2)$$

Thus we are able to improve upon $(s_1^*, s_2)$ and hence $(s_1^*, s_2)$ is not a Nash equilibrium.

Suppose, $s_1^* > s_2^*$. Then

$$u_1(s_1^*, s_2^*) = s_1^*$$
$$u_2(s_1^*, s_2^*) = 2s_2^*$$

Now let us choose $s_1 \ni 1 > s_1 > s_1^*$. Then

$$u_1(s_1, s_2^*) = s_1 > s_1^* = u_1(s_1^*, s_2^*)$$

Thus we can always improve upon $(s_1^*, s_2^*)$.

Case 2: $s_1^* < \frac{1}{2}$

In this case, the unique best response for player 2 is 1, i.e.,

$$s_2^* = 1$$

However $(s_1^*, 1)$ cannot be a Nash equilibrium since any $s_1$ such that $s_1^* < s_1 < 1$ is such that

$$u_1(s_1, 1) = 2s_1$$
$$> 2s_1^*$$
$$= u_1(s_1^*, 1)$$

Interpretations of Nash Equilibrium

Interpretation of Nash equilibrium is one of the most extensively discussed and debated topics in game theory. Very many interpretations have been provided.

Note that a Nash equilibrium is a profile of strategies, one for each of the $n$ players, that has the property that each player’s choice is his best response to the choices of the other $(n - 1)$ players. By deviating from a Nash equilibrium strategy, a player is not going to be better off given the Nash equilibrium strategies of the other players. The following provides several interpretations put forward by game theorists.
Interpretation 1: Prescription

An adviser or a consultant to the $n$ players would basically advise a Nash equilibrium strategy profile to the players.

- If the adviser prescribes strategies that do not constitute a Nash equilibrium, then some players would find that it would be better for them to do differently than advised.
- If the adviser prescribes strategies that do constitute a Nash equilibrium, then all players are kept happy because there don’t have to deviate from the strategies.

Thus a logical, rational, “good” adviser will advise Nash equilibria.

Interpretation 2: Prediction

If the players are rational and intelligent, then a Nash equilibrium is a good prediction for the game.

- For example, iterated elimination of strongly dominated strategies will lead to a reduced form which will include a Nash equilibrium.
- In many games, iterated elimination of strongly dominated strategies leads to a unique prediction, the Nash equilibrium.

Interpretation 3: Self-Enforcing Agreement

A Nash equilibrium can be viewed as an implicit or explicit agreement between the players. Once this agreement is reached, it does not need any external means of enforcement because it is in the self-interest of each player to follow this agreement if the others do.

- In a non-cooperative game, since agreements cannot be enforced, Nash equilibrium agreements are the only ones sustainable.

Interpretation 3: Evolution and Steady-State

A Nash equilibrium is a potential stable point of a dynamic adjustment process in which players adjust their behavior to that of other players in the game, constantly searching for strategy choices that will give them the best results.

- This argument is used to explain biological evolution.
- In this interpretation, Nash equilibrium is the outcome that results over time when a game is played repeatedly.
- Nash equilibrium is like a stable social convention that people are happy to maintain forever.

Focal Point Effect

- If a game has multiple Nash equilibria, then an important question to ask is which of these would be implemented by the players?
- This question was investigated by Schelling (Nobel Prize winner in 2005) who proposed the focal point effect.
According to Schelling, anything that tends to focus the player’s attention on one equilibrium may make them all expect it and hence fulfill it, like a self-fulfilling prophecy.

Such a Nash equilibrium, which has some property that distinguishes it from all other equilibria is called a **focal equilibrium**.

**Example**

Recall the BOS game

\[
\begin{array}{c|cc}
& 1 & 2 \\
\hline
M & 2,1 & 0,0 \\
F & 0,0 & 1,2 \\
\end{array}
\]

Here \((M, M)\) and \((F, F)\) are both Nash equilibria. If an extremely popular music concert is announced, then \((M, M)\) may become the focal equilibrium. On the other hand, if an extremely popular film is running in town and everybody is talking about it, then \((F, F)\) may become the focal equilibrium.

**To Probe Further**

The material discussed in this chapter draws upon mainly from the the books by Myerson [4] and Osborne and Rubinstein [5].

The books by Osborne [6], Straffin [7], and Binmore [8] contain very interesting discussion on Nash equilibrium.

As is well known, the notion of Nash equilibrium was proposed by John Nash as part of his doctoral work which was published in [1, 2]. Holt and Roth [9] have recently published an insightful perspective on the notion of Nash equilibrium.

**References**


