Game Theory

Lecture Notes By

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COOPERATIVE GAME THEORY
Coalitional Games: Introduction

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

In the previous chapter, we have studied the two person bargaining problem where we have explored the effect of cooperation between two players. In this chapter, we introduce multiplayer coalitional games.

1 Multi-Person Bargaining Problem

Let $N = \{1, \ldots, n\}$ be the set of players. We have already seen the Nash bargaining solution for a two player game. What will this solution look like for an n-player game with $n > 2$? Let F be the set of feasible allocations that the players can get if they all work together. Let us assume that F is a closed convex subset of $\mathbb{R}^n$. Let $(v_1, \ldots, v_n)$ be the disagreement payoff allocation the players would expect if they did not cooperate. Also assume that the set $\{(y_1, \ldots, y_n) \in F : y_i \geq v_i \ \forall i \in N\}$ is non-empty and bounded. The pair $(F, (v_1, \ldots, v_n))$ is then called an *n-person bargaining problem*. The bargaining problem $(F, (v_1, \ldots, v_n))$ is said to be *essential* if there exists $y \in F$ such that $y_i > v_i \ \forall i \in N$.

Suppose $(F, (v_1, \ldots, v_n))$ is essential. Then its Nash bargaining solution can be defined to be the unique strongly efficient allocation vector that maximizes

$$\prod_{i \in N}^{n} (x_i - v_i)$$

over all vectors $x \in F$ such that $x_i \geq v_i \ \forall i \in N$.

However this Nash bargaining solution ignores the possibility of cooperation among subsets of the players as shown in the series of four examples below. Consequently for $n > 2$, Nash bargaining solution may not give a credible solution. So we have to look for more appropriate solution concepts.

1.1 Divide the Dollar Problem

Example 1: Divide the Dollar Game - Version 1

Here there are three players, so $N = \{1, 2, 3\}$. The players wish to divide a total wealth of 300 (real number) among themselves. Each player can propose a payoff such that no player's payoff is negative and the sum of all the payoffs does not exceed 300. The strategy sets can therefore be defined as follows:

$$S_1 = S_2 = S_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 300;\ \ x_1 \geq 0;\ x_2 \geq 0,\ x_3 \geq 0\}$$

Assume that the players will get 0 unless all three players propose the same allocation. That is for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Note in this game that the players can achieve any allocation in which their payoffs are non-negative and sum to ≤ 300. The minimum guaranteed wealth is 0 for each player. The above game can therefore be described as a three person bargaining problem (F, v) where:

$$\begin{aligned} F &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \leq 300, x_1 \geq 0;\ x_2 \geq 0,\ x_3 \geq 0\} \\ v &= (v_1, v_2, v_3) = (0, 0, 0) \end{aligned}$$

The Nash bargaining solution for this problem is $x = (100, 100, 100)$, which clearly is a reasonable outcome for this situation.

Example 2: Divide the Dollar Game - Version 2

This is a variation of Version 1 with the difference that players get 0 unless player 1 and player 2 propose the same allocation in which case the allocation proposed by players 1 and 2 is enforced. That is, for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The same bargaining problem (F, v) as in Version 1 would describe the situation here and hence the Nash bargaining solution for this problem also is $x = (100, 100, 100)$. This solution looks unreasonable because players 1 and 2 together determine the payoff allocation and player 3 is not involved in the decision. So, we would expect players 1 and 2 to divide the payoff equally between them, leading to the allocation $(150, 150, 0)$. Another viewpoint which supports this argument is as follows. Suppose 1 and 2 ignore 3 and play out a two person cooperative game. The resulting two person game would have the Nash bargaining solution that divides 300 equally between 1 and 2.

Having noted the above, there are a few reasons for arguing in favor of the solution $(100, 100, 100)$ for this problem.

- The players are required to choose their proposals simultaneously and both $(100, 100, 100)$ and $(150, 150, 0)$ are equilibria for the players.

- Even if non-binding preplay communication is possible between the players, there is still an equilibrium in which both players 1 and 2 expect each other to ignore anything that was said. This again points to $(100, 100, 100)$ as a possibility.

- If player 3 has any influence, then player 3, being rational, would clearly try to influence players 1 and 2 to go for the equilibrium $(100, 100, 100)$.

However, there is one central assumption based on which the outcome $(100, 100, 100)$ loses credibility and fails any justification. This is the *effective negotiation* assumption which is a natural assumption to make, as articulated by Myerson [1]. The members of a coalition of players are said to *negotiate effectively* and are said to form an *effective coalition* if the players, on realizing that there is a feasible change in their strategies that would benefit them all, would all agree to actually make such a change unless such a change contradicts some agreements that some members of the coalition might have made with other players outside this coalition, in the context of some other equally effective coalition. According to Myerson [1], effective negotiation is the key assumption that distinguishes cooperative game theory from non-cooperative game theory.

The n-person Nash bargaining solution would be relevant if the only coalition that can negotiate effectively is the *grand coalition* that includes the whole of N. If other coalitions also can negotiate effectively, then the Nash solution is no longer relevant. This is because it ignores all information about the power of multi-player coalitions other than the grand coalition N. In Version 1 of the divide-the-dollar game, no coalition that is smaller than $\{1, 2, 3\}$ can guarantee more than 0 to its members. In Version 2, the coalition $\{1, 2\}$ could guarantee its members any payoff allocation that they could get in $\{1, 2, 3\}$.

Example 3: Divide the Dollar Game - Version 3

This is a slight variation of Version 2 with the difference that players get 0 unless player 1 and player 2 propose the same allocation or player 1 and player 3 propose the same allocation, in which case they would get what is proposed. That is, for $i = 1, 2, 3$,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_1 = s_2 = (x_1, x_2, x_3) \text{ or } s_1 = s_3 = (x_1, x_2, x_3) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

The bargaining problem (F, v) as in Versions 1 and 2 would describe the situation here and hence the Nash bargaining solution for this problem also is $x = (100, 100, 100)$. Much like in the case of Version 2, this solution also looks quite unreasonable since players 1 and 2 together or players 1 and 3 together determine the payoff allocation. Player 1 is necessarily involved in both the above situations. So, we would expect the players to divide the payoff in a way that players 2 and 3 get the same payoff but this payoff should be less than the payoff that player 1 would get (since player 1 has to necessarily agree for a non-zero allocation). This leads to uncountably infinite number of possibilities, such as (120, 90, 90), (150, 75, 75), (200, 50, 50), (280, 10, 10), etc. One can even suggest an allocation (300, 0, 0) on the ground that player 1 is indispensable for a non-zero allocation.

Example 4: Divide the Dollar Game - Version 4 (Majority Voting Game)

In this version, the players get 0 unless there is some pair of players $\{1,2\}$, $\{2,3\}$, or $\{1,3\}$ who propose the same allocation, in which case they get this allocation. That is,

$$\begin{aligned} u_i(s_1, s_2, s_3) &= x_i \quad \text{if } s_j = s_k = (x_1, x_2, x_3) \text{ for some } \ j \neq k \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Here again, the Nash bargaining solution is $(100, 100, 100)$. Clearly, this is perfectly justified for this version because of symmetry and equal bargaining power of the players. Observe that this allocation is a Nash equilibrium. If we assume that every coalition can negotiate effectively, the analysis becomes quite interesting as seen below.

- If players 1 and 2 negotiate effectively in the coalition $\{1,2\}$, they can agree to the allocation $(150, 150, 0)$ which is attractive for both of them. Observe that this allocation is also a Nash equilibrium.

- If $(150, 150, 0)$ is the expected outcome, then player 3 would be eager to persuade player 1 or player 2 to form an effective coalition with him. For example, player 3 would be willing to negotiate an agreement with player 2 to both propose $(0, 225, 75)$. This allocation is also a Nash equilibrium.

- If $(0, 225, 75)$ were to be the expected outcome in the absence of further negotiations, then player 1 would be willing to negotiate an agreement with player 3 to propose an allocation that is better for both of them, say, $(113, 0, 187)$. This allocation is again a Nash equilibrium.

- It turns out that in any equilibrium of this game, there is always at least one pair of players who would both do strictly better by jointly agreeing to change their strategies together.

The above sequence of coalitional negotiations will have no end. There are two possible ways in which the negotiations could conclude.

1. Let us say that a player, having negotiated an agreement as part of a coalition, cannot later negotiate a different agreement with another coalition, that does not contain all the members of the first coalition. For example, if the grand coalition $\{1,2,3\}$ negotiated the agreement $(100, 100, 100)$ before any two player coalition could negotiate separately, then no two player coalition can veto this outcome. Also, if players 1 and 2 first negotiated an agreement $(150, 150, 0)$, then player 3 would be unable to increase this payoff by negotiating with player 1 or player 2 separately. It is clear that the order in which coalitions can negotiate may crucially determine the outcome of the game. The advantage lies with coalitions that negotiate earlier.

2. Suppose the negotiated agreements are tentative and non-binding. Thus a player who negotiates in a sequential manner in various coalitions can nullify his earlier agreements and reach a different agreement with a coalition that negotiates later. Here the order in which negotiations are made and nullified will have a bearing on the final outcome. For example, let us say the order of negotiations is $\{1,2\}$, $\{2,3\}$, $\{1,3\}$ and $\{1,2,3\}$. Here any agreement by $\{1,2\}$ and $\{2,3\}$ in that order to pay non-zero amount to player 2 can be overturned by the coalition $\{1,3\}$ which might agree on $(150, 0, 150)$. Player 2 may not be able to make them concede anything when the turn of coalition $\{1,2,3\}$ arises. As another example, assume that player 1 believes that,

any negotiated agreement with player 2, would be overturned by player 3. Player 1 may first suggest $(100, 100, 100)$ and stick to it and refuse to agree for $(150, 150, 0)$. This he would do to prevent any possibility of his getting zero payoff. It is clear that coalitions that get to negotiate later hold the advantage in this scheme.

In realistic cooperative scenarios where different coalitions could form, the number of possibilities could be mind-boggling and a systematic analysis of all scenarios may become infeasible. There are as many as $2^n - 1$ such coalitions possible and therefore there is a need for theories of cooperative games that can provide a clear sense of what to expect as a result of the balance of power among various coalitions. Such order-independent theories will be extremely useful but will pose challenges in interpretation because ordering is often natural and important.

2 Games in Characteristic Form (TU Games)

The assumption of *transferable utility* makes cooperative games somewhat tractable. This assumption implies that there is a commodity called money that the players can freely transfer among themselves such that any player's payoff increases by one unit for every unit of money that he gets. With the assumption of transferable utility in place, the cooperative possibilities of a game can be described by a *characteristic function* $v : 2^N \to \mathbb{R}$, that assigns a number $v(C)$ to every coalition $C \subseteq N$. $v(\emptyset)$ is always taken to be zero. $v(C)$ is called the *worth* of the coalition C and it captures the total amount of transferable utility that the members of C could earn without any help from the players outside of C.

Definition: A cooperative game with transferable utility is defined as the pair (N, v) where $N = \{1, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function, with $v(\emptyset) = 0$. We call such a game also as a *game in coalition form*, *game in characteristic form*, or *coalitional game* or *TU game*.

Note that, under the assumption of transferable utility, specifying a single number for each coalition is enough to describe what allocations of utility can be achieved by the members of the coalition.

Non-Transferable Utility (NTU) Games

In contrast, games without transferable utility (also called NTU coalitional games or games in NTU coalitional form) are defined as follows.

Definition: An NTU coalitional game on the set of players N is any mapping $V(.)$ on the domain 2^N such that, for any coalition $C \subset N$,

- $V(C)$ is a non-empty closed and convex sub set of $\mathbb{R}^{|C|}$, and
- The set $\{x : x \in V(C) \text{ and } x_i \geq v_i \ \forall i \in C\}$ is a bounded subset of $\mathbb{R}^{|C|}$, where

$$\begin{aligned} v_i &= \max\{y_i : y \in V(\{i\})\} \\ &< \infty \ \forall i \in N \end{aligned}$$

Here $V(C)$ is the set of expected payoff allocations that the members of coalition C could guarantee for themselves if they act cooperatively. An NTU game is a generalization of a TU game.

In the remainder of the discussion, we will consider only TU games.

2.1 Examples of TU Games

2.1.1 Characteristic Functions for Divide the Dollar Games

The Divide-the-Dollar - Version 1 game discussed in Example 1 has the following characteristic function.

$$\begin{aligned} v(\{1,2,3\}) &= 300 \\ v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

Version 2 of the game has the characteristic function:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = 300 \\ v(\{2,3\}) &= v(\{1,3\}) = 0 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

Version 3 of the game has the characteristic function:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = v(\{1,3\}) = 300 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0 \end{aligned}$$

Version 4 of the game (majority voting game) has the characteristic form:

$$\begin{aligned} v(\{1,2,3\}) &= v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 300 \\ v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \end{aligned}$$

2.1.2 A Voting Game

This example is taken from [2]. Consider that the Parliament of a certain Nation has four political parties 1, 2, 3, 4 with 45, 25, 15, 12 members respectively. To pass any bill, at least 51 votes are required. This situation could be modeled as a TU games with $N = \{1,2,3,4\}$ and

$$v(1) = v(2) = v(3) = v(4) = 0$$

$$v(12) = v(13) = v(14) = v(123) = v(124) = v(134) = v(234) = v(1234) = 100$$

$$v(23) = v(24) = v(34) = 0$$

2.1.3 Minimum Spanning Tree Game

This example is also taken from [2]. Suppose a group of users are to be connected to an expensive resource managed by a central facility (for example, a power plant, a synchrotron, a radio telescope, a high performance cluster, etc.). In order to make utilize this resource, a user should either be directly connected to the facility or be connected to some other connected user. Figure 1 provides a picture of a typical network of customers. Here the set of players is the set of all users and $v(C)$ for any coalition of users is the cost of connecting all users in C directly to the facility minus the cost of a minimum cost spanning tree that spans the facility and the users in C.

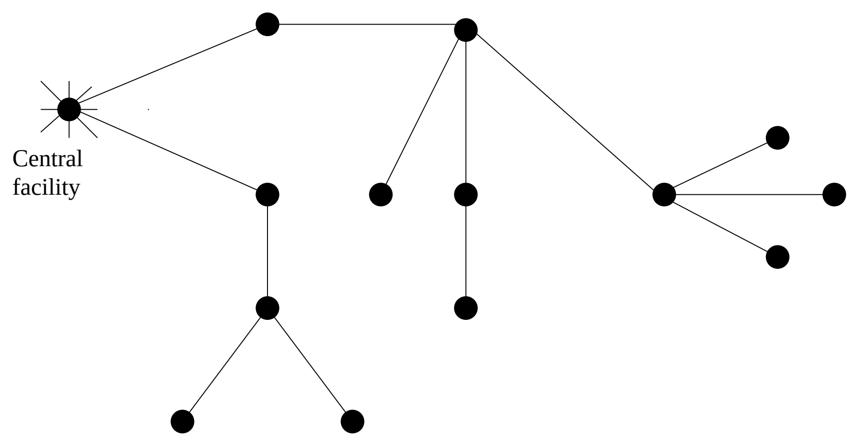

Figure 1: A network of users connected to a critical resource (central facility)

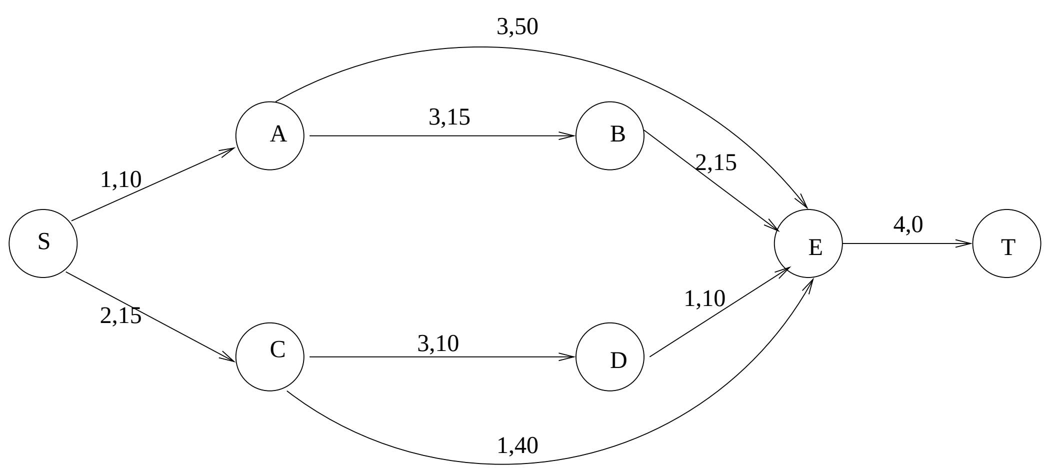

Figure 2: A logistics network

2.1.4 A Logistics Game

Figure 2 shows a logistics network that provides connectivity between two important cities S and T. There are five logistics hubs A, B, C, D, E which are intermediate points from S to T. The transportation is provided by service providers 1, 2, 3, 4. Each edge in the network is labeled by two quantities namely the service provider and the cost of service. For example, the label 3, 15 on the directed edge from A to B means that service provider 3 provides the logistics service from A to B at a cost of 15 units. Assume that movement from S to T fetches a revenue of 100 units. The objective is to choose an optimal path from S to T that minimizes the cost of moving from S to T. We can formulate this as a cooperative game with $N = \{1, 2, 3, 4\}$ and with characteristic function

$$v(1) = v(2) = v(3) = v(4) = 0$$

$$v(12) = v(13) = v(14) = v(23) = v(24) = v(34) = v(234) = v(123) = 0$$

$$v(134) = 100 - 60 = 40$$

$$v(124) = 100 - 55 = 45$$

$$v(1234) = 100 - 35 = 65$$

3 Representations for TU Games

Like in the case of Nash bargaining problems, there are several different ways in which the characteristic function could be defined for TU games, starting from the base model, namely a strategic form game. Three of the more common representations are:

1. Minimax representation
2. Defensive equilibrium representation
3. Rational threats representation

We describe the minimax representation below and postpone a discussion of the other two representations to the Appendix.

3.1 Minimax Representation

Let $\Gamma = \langle N, (S_i), (u_i) \rangle$ be a n-person strategic form game with transferable utility. Let $C \subset N$ be any coalition of players. $N \setminus C$ is the set of all players who are not in the coalition C. Let

$$S_{N \setminus C} = \times_{j \in N \setminus C} \; S_j$$

$$S_C = \times_{j \in C} \; S_j$$

Now, $\Delta(S_C)$ is the set of correlated strategies available to coalition C. Let $u_i(\sigma_C, \sigma_{N \setminus C})$ denote player i's expected payoff, before transfers of money, when the correlated strategies σ_C and $\sigma_{N \setminus C}$ are independently implemented:

$$u_i(\sigma_C, \sigma_{N \setminus C}) = \sum_{s_C \in S_C} \; \sum_{s_{N \setminus C} \in S_{N \setminus C}} \sigma_C(s_C) \sigma_{N \setminus C}(s_{N \setminus C}) \; u_i(s_C, s_{N \setminus C})$$

It was suggested by von Neumann and Morgenstern that the characteristic function should be defined by:

$$v(C) = \min_{\sigma_{N\setminus C} \in \Delta(S_{N\setminus C})} \max_{\sigma_C \in \Delta(S_C)} \sum_{i \in C} u_i(\sigma_C, \sigma_{N\setminus C})$$

$v(C)$ may be interpreted as the maximum sum of utility payoffs that the members of coalition C can guarantee themselves against the most offensive threat by the complementary coalition. This is called the minimax representation in coalitional form of the strategic form game Γ with transferable utility.

3.1.1 Intuition on Minimax Representation

This representation implicitly assumes that a coalition C should be concerned that $N \setminus C$ would attack C offensively if the members of C decided to cooperate with each other but without the players in $N \setminus C$. However, offensively minimizing the sum of payoffs of the players in C may not be in the best interests of players in $N \setminus C$. Note that the primary interest of all players is to maximize their own payoffs. A justification for the assumption that the members of $N \setminus C$ might act offensively against C is as follows. When all players ultimately cooperate together as a part of the grand coalition N and the players are negotiating over the possible division of worth $v(N)$, the players in $N \setminus C$ can jointly commit themselves to an offensive threat that would be carried out only in the improbable event that the players in C break off negotiations with players in $N \setminus C$. Such a threat by $N \setminus C$ is a deterrent on the coalition C and the members of C would be willing to concede a larger share to $N \setminus C$.

3.1.2 An Example for Minimax Representation of a TU Game

This example is taken from the book by Straffin [3]. Consider a strategic form game with $N = \{1, 2, 3\}$, $S_1 = S_2 = S_3 = \{A, B\}$, and with payoffs as shown in Table 1.

(s_1, s_2, s_3)	$u_1(s_1, s_2, s_3)$	$u_2(s_1, s_2, s_3)$	$u_3(s_1, s_2, s_3)$
(A, A, A)	1	1	-2
(A, A, B)	3	-2	-1
(A, B, A)	-4	3	-1
(A, B, B)	-6	-6	12
(B, A, A)	2	-4	2
(B, A, B)	2	2	-4
(B, B, A)	-5	-5	10
(B, B, B)	-2	3	-1

Table 1: Payoff matrix for the given strategic form game

If players 2 and 3 cooperate and decide to move jointly against player 1, then we have a virtual game with two players $\{1\}$ and $\{2, 3\}$. Then using minmax representation, the payoffs will be as shown in Table 2.

The game shown in Table 2 is a zero sum game with optimal strategies for the players as: $(\frac{3}{5}, \frac{2}{5})$ for player 1; (0, 1) for player 2 and $(\frac{4}{5}, \frac{1}{5})$ for player 3. this leads to a payoff of -4.4 for player 1 and a payoff of 4.4 for the coalition $\{2, 3\}$. Likewise, the minimax values can be computed for all possible coalitional structures and this yields the following characteristic function.

{1}	{2,3}			
	(A,A)	(B,A)	(A,B)	(B,B)
A	1, -1	-4, 4	3, -3	-6, 6
B	2, -2	-5, 5	2, -2	-2, 2

Table 2: Payoff matrix with two coalitions $\{1\}$ and $\{2,3\}$

$$\begin{aligned} v(1) &= -4.4;\ v(2) = 4;\ v(3) = -1.43 \\ v(23) &= 4.4;\ v(13) = 4;\ v(12) = 1.43 \\ v(123) &= 0 \end{aligned}$$

4 Superadditive Games

Definition: A TU game (N, v) is said to be *superadditive* if

$$v(C \cup D) \geq v(C) + v(D) \quad \forall\, C, D, \subseteq N \text{ such that } C \cap D = \phi$$

Intuitively, the value of union of two disjoint coalitions is higher than the sum of values of the two parts, that is, two disjoint coalitions on coming together produce an additional value beyond the sum of the individual values. It can be shown that all games (N, v) obtained from strategic form games using minimax representation will satisfy this property. The defensive equilibrium and rational threats representations do not necessarily lead to super-additive games.

4.1 Examples for Superadditivity

Example 1

The majority voting game with $N = \{1, 2, 3, 4\}$ and v given by $v(1) = v(2) = v(3) = 0$ and $v(12) = v(13) = v(23) = v(123) = 300$ is a superadditive game.

Example 2

The following game (called the communication satellite game [3]) is also superadditive. $v(1) = 3;\ v(2) = 2;\ v(3) = 1$
$v(12) = 8;\ v(13) = 6.5;\ v(23) = 8.2$
$v(123) = 11.2$

Example 3

The following three player game is not superadditive. $v(1) = 10;\ v(2) = 15;\ v(3) = 20$
$v(12) = 20;\ v(13) = 30;\ v(23) = 35$
$v(123) = 40$

4.2 Superadditive Cover

Definition: Given a game (N, v) in coalitional form, the *super-additive cover* of v is the super-additive game (N, w) in coalitional form with the lowest possible worths for all coalitions such that

$$w(C) \geq v(C) \;\; \forall C \subset N$$

Let $P(C)$ be the set of all partitions of C. The super-additive cover w of the game v in coalitional form satisfies the equation

$$w(C) = \max \left\{ \sum_{j=1}^{k} v(T_j) : \{T_1, \dots, T_k\} \in P(C) \right\} \;\; \forall C \subset N$$

The above definition implies that the worth of a coalition in the super-additive cover is the maximum worth that the coalition could achieve by breaking up into a set of smaller disjoint coalitions. The notion of a super-additive cover provides a way to define a super-additive game corresponding to any game in coalitional form.

4.3 Imputations

Given a TU game (N, v), an imputation is an allocation $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ that satisfies

- Individual Rationality : $x_i \geq v(\{i\}) \quad \forall\, i \in N$
- Collective Rationality : $\sum_{i \in N} x_i = v(N)$

An imputation keeps all individual players happy and also distributes the total value of the grand coalition among the players (Pareto efficiency).

4.4 Essential and Inessential Games

A superadditive game (N, v) is said to be inessential if

$$\sum_{i \in N} v(i) = v(N)$$

and essential if

$$\sum_{i \in C} v(i) \leq v(N)$$

If (N, v) is inessential then,

$$\sum_{i \in N} v(i) = v(C) \quad \forall\, C \subseteq N$$

Therefore the only possible imputation for an inessential game is $(v(1), v(2), \dots, v(n))$. On the other hand, there are infinitely many imputations possible for an essential game.

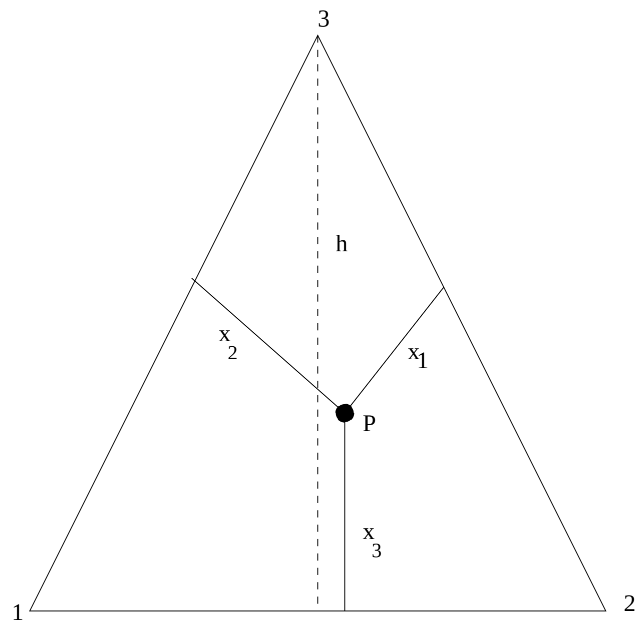

Figure 3: An equilateral triangle, $x_1 + x_2 + x_3 = h$

4.5 Strategic Equivalence of TU Games

Two TU games (N, v) and (N, w) are said to be strategically equivalent if there exist constants $c_1, c_2, \ldots, c_n$ and $b > 0$ such that

$$w(c) = b(v(C) + \sum_{i \in C} c_i) \quad \forall\, C \subseteq N$$

Intuitively, strategic equivalence means that the dynamics among the players would be identical in the two games. An important result concerning strategic equivalence is that any superadditive, essential n-person characteristics form game G in the strategically equivalent to a unique game with

$$N = \{1, 2, \ldots, n\}$$

$$v(1) = v(2) = \cdots = v(n) = 0i; \quad v(N) = 1$$

$$0 \leq v(C) \leq 1 \quad \forall\, C \subseteq N$$

This unique game is called the *0-1 normalization* of the original game.

4.6 Triangular Representation for Three Person Superadditive Games

The imputations in any three person game with $v(1) = v(2) = v(3) = 0$ and $v(N) = 1$ can be represented using an interesting triangular representation. This representation uses the following property: Suppose P is any point in a equilateral triangle with height h. Then the sum of the distances from P to the three sides of the triangle is equal to h (see Figure 3).

If we consider the majority voting game with $v(1) = v(2) = v(3) = 0; v(12) = v(13) = v(23) = v(123) = 1$ then any point $\rho = (x_1, x_2, x_3)$ represents an imputation because $x_1 \geq 0;\ x_1 \geq 0;\ x_2 \geq 0;\ x_3 \geq 0;\ x_1 + x_2 + x_3 = 1$. Figure 4 depicts several representative imputations.

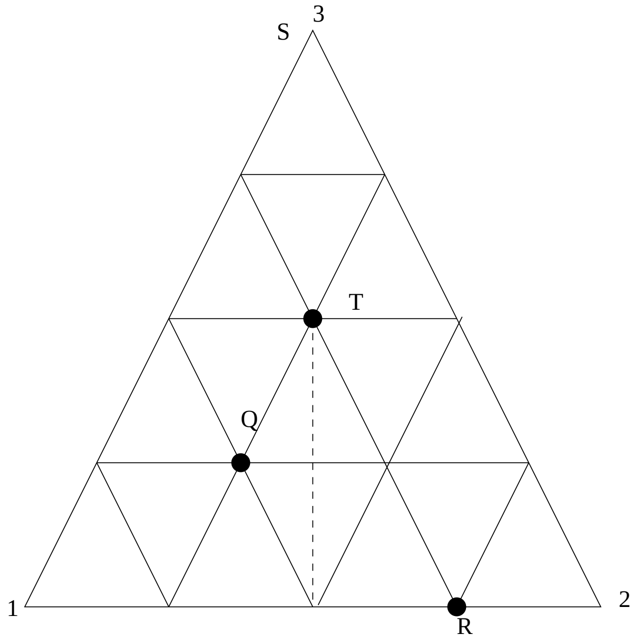

Figure 4: $Q = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$; $R = (\frac{1}{4}, \frac{3}{4}, 0)$; $S = (0, 0, 1)$; $T = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are a few imputations

4.6.1 Example: Bargaining in Majority Voting Game

Consider the majority voting game (Divide-the-Dollar-version 4) where we have $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0; v(12) = v(13) = v(23) = v(123) = 300$. Suppose we start with an allocation $(150, 150, 0)$ which indicates an alliance between players 1 and 2. Player 3 can entice player 1 by proposing an allocation such as (180, 0, 120). Player 2 can now propose an allocation such as (0,120,180) and draw player 3 out of alliance with player 1. In this particular game, bargaining can go on endlessly without any stable allocation being agreed upon. A graphical representation of this endless negotiation is presented in Figure 5.

4.7 Domination of Imputation

An imputation $x = (x_1, \ldots, x_n)$ of a TU game (N, v) is said to dominate an imputation $y = (y_1, \ldots, y_n)$ if there exists a coalition C such that

$$\sum_{i \in C} x_i \leq v(C)$$

$$x_i > y_i \quad \forall\, i \in C$$

We make several observations about the notion of domination.

1. Given two imputations x and y, it is possible that neither dominates the other. An immediate example would be three person majority voting game where the imputation $x = (150, 150, 0)$ neither dominates nor is dominated by the imputation $y = (0, 150, 150)$.

2. The relation of domination is not transitive and cycles of domination are possible. Again an immediate example would be majority voting game where imputation (0, 180, 120) dominates

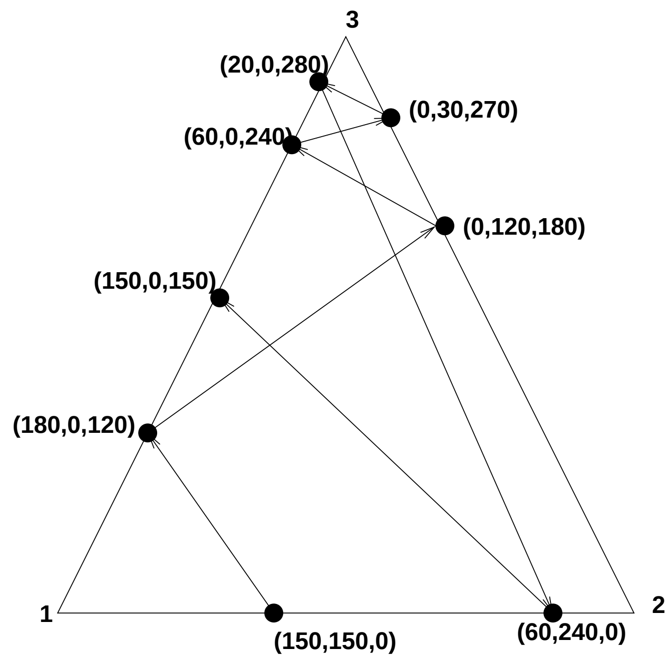

Figure 5: Chain of unending negotiations in majority voting game

imputation (150, 150,0) which dominates imputation (90, 0, 210) which in turn dominates (0, 180, 120).

3. In a TU game, it is possible that every imputation is dominated by some other imputation as shown in Figure 6. Note that the imputation x is dominated by imputations in the shaded regions and dominates imputations in the dotted regions.

To Probe Further

The material in this chapter is mostly based on the books by Myerson [1] and Straffin [3].

Problems

1. Show that a TU game obtained from a strategic form game using the minimax representation is superadditive.

2. (Straffin 1993) [3]. Show that any superadditive, essential, two person TU game (N, v) is strategically equivalent to a unique game (N, w) where

$$w(1) = w(2) = \ldots = w(n) = 0$$

$$w(N) = 0$$

$$0 \leq w(C) \leq 1 \;\; \forall C \subseteq N$$

The game (N, w) is called the 0-1 normalization of (N, v).

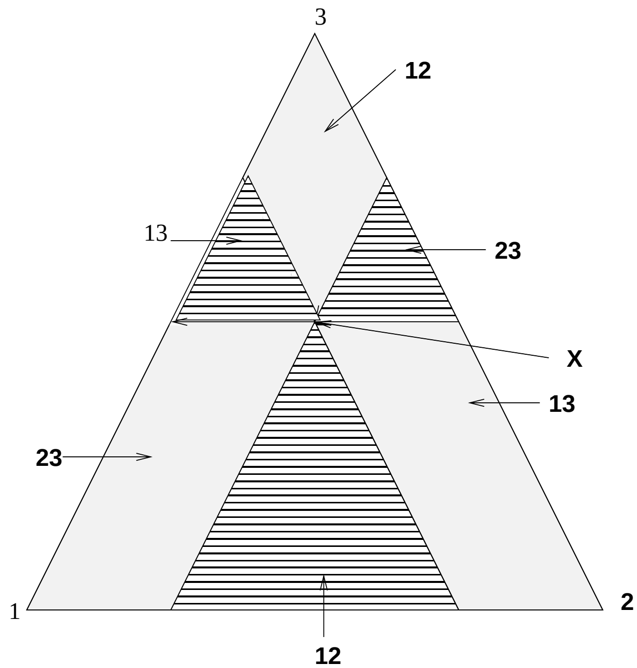

Figure 6: Imputation x is dominated by some imputations and dominates the other imputations

3. (Straffin 1993) [3]. Show that the 0-1 normalization of any three person constant sum game (that is, a TU game where, for every coalition $C \subseteq N$, $v(C) + v(N \setminus C)$ is a constant) is the three person majority voting game.

5 Appendix: Defensive Equilibrium Representation and Rational Threats Representation

5.1 Defensive Equilibrium Representation

Here we assume that complementary coalitions would play an essentially defensive pair of equilibrium strategies against each other. The implicit assumption that each coalition makes here is that the complementary coalition will play an equilibrium strategy and the coalition settles for a defensive strategy by playing *its* equilibrium strategy. For all $C \subset N$, define $\bar{\sigma}_C$ as a correlated strategy belonging to the set

$$\underset{\sigma_C \in \Delta(S_C)}{\text{argmax}} \sum_{i \in C} u_i(\sigma_C, \bar{\sigma}_{N \setminus C})$$

Similarly, define $\bar{\sigma}_{N \setminus C}$ a correlated strategy belonging to the set

$$\underset{\sigma_{N \setminus C} \in \Delta(S_{N \setminus C})}{\text{argmax}} \sum_{j \in N \setminus C} u_j(\bar{\sigma_C}, \sigma_{N \setminus C})$$

Define the characteristic function as

$$v(C) = \sum_{i \in C} u_i(\bar{\sigma_C}, \bar{\sigma}_{N \setminus C})$$

$$v(N \setminus C) = \sum_{j \in N \setminus C} u_j(\bar{\sigma_C}, \bar{\sigma}_{N \setminus C})$$

Then v is called a defensive equilibrium representation in coalition form of the strategic form game Γ with transferable utility.

5.2 Rational Threats Representation

This was proposed by Harsanyi in 1963. This representation is derived by generalizing the rational threats criterion of Nash. Let $C \subset N$ be any coalition. Define $\bar{\sigma}_C$ as a correlated strategy belonging to the set

$$\overset{\text{argmax}}{\sigma_C \in \Delta(S_C)} \left[\sum_{i \in C} u_i(\sigma_C, \bar{\sigma}_{N \setminus C}) - \sum_{j \in N \setminus C} u_j(\sigma_C, \bar{\sigma}_{N \setminus C}) \right]$$

Similarly, define $\bar{\sigma}_{N \setminus C}$ a correlated strategy belonging to the set

$$\overset{argmax}{\sigma_{N \setminus C} \in \Delta(S_{N \setminus C})} \left[\sum_{j \in N \setminus C} u_j(\bar{\sigma_C}, \sigma_{N \setminus C}) - \sum_{i \in C} u_i(\bar{\sigma}_C, \sigma_{N \setminus C}) \right]$$

Now define

$$v(C) = \sum_{i \in C} u_i(\bar{\sigma_C}, \bar{\sigma}_{N \setminus C})$$

$$v(N \setminus C) = \sum_{j \in N \setminus C} u_j(\bar{\sigma_C}, \bar{\sigma}_{N \setminus C})$$

5.3 Some Observations

Note that in all the three representations, the worth of the grand coalition N is the same:

$$v(N) = \max_{s_N \in S_N} \sum_{i \in N} u_i(s_N)$$

The distinction between the three representations can be interpreted in terms of different assumptions about the ability of the coalitions to commit themselves to offensive and defensive threats.

Example: Different Representations of TU Games

Let $N = \{1, 2, 3\}$; $S_i = \{a_i, b_i\}$ for $i = 1, 2, 3$. Suppose the payoff matrix is as shown below.

	$S_2 \times S_3$			
S_1	a_2, a_3	b_2, a_3	a_2, b_3	b_2, b_3
a_1	4, 4, 4	2, 5, 2	2, 2, 5	0, 3, 3
b_1	5, 2, 2	3, 3, 0	3, 0, 3	1, 1, 1

a_i is to be interpreted as a *generous strategy* and b_i as a *selfish strategy*.

1. In the minimax representation, each coalition C gets the most that it could guarantee itself if the players in the complementary coalition were selfish.

$$
\begin{aligned}
v(\{1,2,3\}) &= 12 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4 \\
v(\{1\}) &= v(\{2\}) = v(\{3\}) = 1
\end{aligned}
$$

2. In the defensive equilibrium representation, the members of a two player coalition can actually increase the sum of their payoffs by both being generous. Here the characteristic function would be

$$
\begin{aligned}
av(\{1,2,3\}) &= 12 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 4 \\
v(\{1\}) &= v(\{2\}) = v(\{3\}) = 5
\end{aligned}
$$

   This representation imputes an advantage to a player who acts selfishly alone against a generous two player coalition.

3. In the rational threats representation, both the offensive and defensive considerations are taken into account. Here all coalitions smaller than N choose selfishness in the game.

$$
\begin{aligned}
v(\{1,2,3\}) &= 12 \\
v(\{1,2\}) &= v(\{1,3\}) = v(\{2,3\}) = 2 \\
v(\{1\}) &= v(\{2\}) = v(\{3\}) = 1
\end{aligned}
$$

If all three representations coincide, the game is said to have *orthogonal coalitions*. As to which representation is to be used depends on what the modeler or designer perceives as most appropriate. In some cases, as in the orthogonal case, there may be a natural representation that might immediately suggest itself.

References

[1] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.

[2] Yoam Shoham and Kevin Leyton-Brown. *Multiagent systems: Algorithmic, Game-Theoretic, and Logical Foundations.* Cambridge University Press, New York, USA, 2009, 2009.

[3] Philip D. Straffin Jr. *Game Theory and Strategy.* The Mathematical Association of America, 1993.