Game Theory

Lecture Notes By

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Chapter 32. The Shapley Value

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

The Shapley value is a popular solution concept in cooperative game theory that provides a unique allocation to a set of players in a coalitional game. Like the Nash bargaining solution, the Shapley value is based on a set of axioms that decide the allocation to a player based on the player's marginal contribution. In this chapter, we present the Shapley axioms and prove the existence and uniqueness of the Shapley value.

Given a coalitional game, the core may be empty or may be very large. This certainly causes difficulties in getting sharp predictions for the game. The *Shapley value* is a solution concept which is motivated by the need to have a solution concept that would predict a unique expected payoff allocation for every given coalitional game. The Shapley value concept was proposed using an axiomatic approach by Shapley in 1953, as a part of his doctoral dissertation at the Princeton University. Given a game in coalitional form (N, v), the Shapley value is denoted by $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$ where $\phi_i(v)$ is the expected payoff to player i. The Shapley value tries to capture how coalitional competitive forces influence the possible outcomes of a game. It describes a reasonable or fair way of dividing the gains from cooperation given the strategic realities captured by the characteristic function. It is a *descriptive, normative* solution concept.

1 Shapley's Axioms

Let (N, v) be a game in coalitional form. Let π be a permutation on the set N. Let $(N, \pi v)$ be the coalitional game such that

$$\pi v(\{\pi(i) : i \in C\}) = v(C) \ \forall C \subseteq N$$

This means that the role of any player $i \in N$, in the game (N, v) is essentially the same as the role of player $\pi(i)$ in $(N, \pi v)$.

Lloyd Shapley can be considered as one of the most influential game theorists of all time. He has made numerous pioneering contributions which have started new areas in game theory. Many concepts, lemmas and theorems have been named after him. These include of course the Shapley value which is perhaps the most popular solution concept in cooperative game theory.

Others include: (1) Bondareva - Shapley theorem which provides a necessary and sufficient condition for the non-emptiness of the core of a coalitional game and which also implies that convex games have non-empty cores (2) Gale - Shapley theorem which provides the first and perhaps the most used solution to the stable marriage problem (3) Aumann - Shapley pricing that pioneered the pricing of products and services that share resources (4) Shapley - Folkmann lemma which settled the question of convexity of addition of sets (5) Shapley-Shubik power index for determining voting power. Moreover, stochastic games were first proposed by Shapley as early as 1953. Potential games which are extensively used by researchers these days were proposed by Shapley and Dov Monderer in 1996. His joint work with Maschler and Peleg on the kernel and the nucleolus is quite path breaking so also his work with Robert Aumann on non-atomic games and on long-term competition.

Shapley was born in 1923 in Cambridge, Massachusetts and he was a genius in mathematics from an early age. He joined Harvard university for his undergraduate studies and got his A.B. Degree in 1948. During 1943 - 45, he worked for the American Military and as a Sargent in Army Corps in 1943, he brilliantly broke the Soviet weather code and was decorated with a Bronze star at the age of 20. He started working with Albert Tucker at the Princeton University in 1949 and got his Ph.D. in 1953. The work on what is now called the Shapley value was carried out during this period. The title of his doctoral dissertation was *Additive and Nonadditive Set Functions*. John Nash and Shapley were contemporaries working with Albert Tucker. From 1954 to 1981, Shapley spent 27 years at the Rand Corporation as research mathematician and produced many path breaking results. Since 1981, he has been at the University of California, Los Angeles, working in the Department of Economics as well as the Department of Mathematics. He is a recipient of the John Von Neumann theory prize and many other prestigious honors.

For example, suppose $N = \{1, 2, 3\}$. Consider the permutation π on N defined by $\pi(1) = 3; \pi(2) = 1; \pi(3) = 2$. Then the game $(N, \pi v)$ will be the following: $\pi v(1) = v(2); \pi v(2) = v(3); \pi v(3) = v(1); \pi v(12) = v(23); \pi v(23) = v(13); \pi v(13) = v(12); \pi v(123) = v(123)$, where $v(123)$ denotes $v(\{1, 2, 3\})$, etc.

Shapley proposed three axioms to describe the desirable properties that we would expect a good solution concept to satisfy.

- Axiom 1 : Symmetry
- Axiom 2 : Linearity
- Axiom 3 : Carrier

1.1 Axiom 1: Symmetry

For any $v \in \mathbb{R}^{2^n-1}$, any permutation π on N, and any player $i \in N$,

$$\phi_{\pi(i)}(\pi v) = \phi_i(v)$$

Informally, Shapley value of a player relabeled by a permutation, under the permuted value function is the same as the Shapley value of the original player under the original value. This axiom implies that only the role of a player in the game should matter. The labels or specific names used in N are irrelevant.

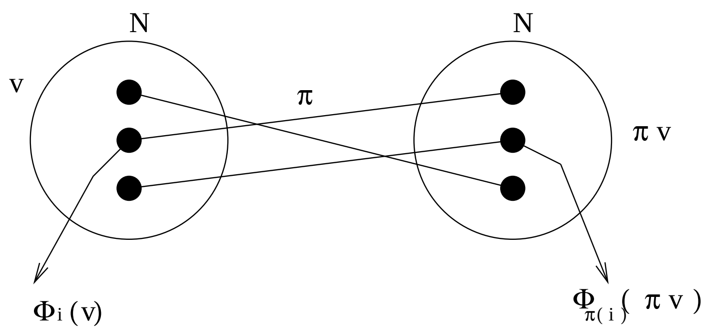

Figure 1: Axiom 1: Symmetry of Shapley value

1.2 Axiom 2: Linearity

Let (N, v) and (N, w) be any two coalitional games. Suppose $p \in [0, 1]$. Define a new coalitional game $(N,\ \ pv + (1-p)w)$ as follows.

$$(pv + (1-p)w)(C) = pv(C) + (1-p)w(C)\ \ \forall C \subseteq N$$

Axiom 2 states that, given any two coalitional games (N, v) and (N, w), any number $p \in [0, 1]$, and any player $i \in N$,

$$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w)$$

In other words, the Shapley value of a player for a convex combination of coalitional games is the convex combination of Shapley values of the player in the individual games. This axiom asserts that the expected payoff to each player is the same before resolution of uncertainty and after resolution of uncertainty.

An Example for Linearity

Let us say there is a sports contest (say a cricket match) where the toss does play an important role. The mapping v corresponds to the coalitional game when the team has won the toss while w corresponds to the situation when the team has lost the toss.

- $\phi_i(pv + (1-p)w)$ is the expected payoff to player i before the toss takes place.
- $p\phi_i(v) + (1-p)\phi_i(w)$ is the expected payoff to player i after the toss takes place.

Linearity in this case means that these two expected payoffs are the same.

1.3 Axiom 3: Carrier

A coalition D is said to be a *carrier* of a coalitional game (N, v) if

$$v(C \cap D) = v(C) \quad \forall C \subseteq N$$

If D is a carrier and $i \notin D$, then

$$v(\{i\}) = v(\{i\} \cap D) = v(\phi) = 0$$

If D is a carrier of (N, v), all players $j \in N \setminus D$ are called *dummies* in (N, v) because their entry into any coalition cannot change the worth of the coalition. Also, for any $C \subseteq N$ and $i \notin D$,

$$v(C \cup \{i\}) = v((C \cup \{i\}) \cap D) = v(C \cap D) = v(C)$$

Intuitively, D includes all influential players (it might also include non-influential players but it will not exclude any influential player). If D is a carrier and $i \in N$, then for any $C \subseteq N$,

$$v(C) = v(C \cap D) = v(C \cap (D \cup \{i\}))$$

Hence, $D \cup \{i\}$ for any $i \in N$ is also a carrier. In fact, the set N is always a carrier. This means that $v(D) = v(N)$. This can also be seen from

$$v(D) = v(D \cap N) = v(N)$$

It is however possible that no proper subset of N is a carrier. The *Carrier Axiom* (Axiom 3) states that, for any $v \in \mathbb{R}^{2^n-1}$ and any coalition D that is a carrier of (N, v),

$$\sum_{i \in D} \phi_i(v) = v(D) = v(N)$$

The carrier axiom immediately implies that

$$\phi_i(v) = 0 \; \forall i \notin D$$

$$\sum_{i \in N} \phi_i(v) = v(N)$$

This axiom asserts that the players in a carrier set should divide their joint worth (which is equal to the worth of the grand coalition) among themselves. This means the dummies are allocated nothing. The above expression also illustrates a key fact that the Shapley value always divides the worth of the grand coalition among the players of the game. This means that Shapley value implicitly assumes the formation of the grand coalition (however some of these players may be dummy players who do not get anything allocated).

2 Shapley's Theorem and Examples

With the above three axioms in place, we are now in a position to state the famous result due to Shapley [1].

Theorem 1 *There is exactly one mapping*

$$\phi : \mathbb{R}^{2^n-1} \to \mathbb{R}^n$$

that satisfies Axiom 1, Axiom 2, and Axiom 3. This mapping satisfies: $\forall i \in N$, $\forall v \in \mathbb{R}^{2^n-1}$,

$$\phi_i(v) = \sum_{C \subseteq N-i} \frac{|C|!\ (n-|C|-1)!}{n!}\ \{v(C \cup \{i\}) - v(C)\}$$

The notation $N - i$ above denotes the set $N \setminus \{i\}$. The term $\frac{|C|!\ (n-|C|-1)!}{n!}$ can be interpreted as the probability that in any permutation, the members of C are ahead of a distinguished player i. The term $v(C \cup \{i\}) - v(C)$ gives the marginal contribution of player i to the worth of the coalition C. Thus the above formula for $\phi_i(v)$ gives the expected contribution of player i to the worth of any coalition.

Suppose there is a collection of n resources and each resource is useful in its own way towards executing a certain service. Suppose $v(N)$ is the total value that this collection of resources would create if all the resources are deployed for the service accomplishment. Let us focus on a certain resource, say resource i. Now, this resource will make a marginal contribution to every subset C of $N-i$ when it is included to the set C. We can choose the set C in $(|C|!\ \ |n-|C|-1|!)$ ways and when this is divided by $n!$, we obtain the probability of choosing a particular subset C. Thus the Shapley value of resource i is the average marginal contribution that resource i will make to any arbitrary coalition that is a subset of $N - i$.

2.1 Example 1: Divide the Dollar Game

First, let us consider Version 3. Recall that $N = \{1, 2, 3\}$; $v(1) = v(2) = v(3) = v(23) = 0$; $v(12) = v(13) = v(123) = 300$. The Shapley value expression for $\phi_1(v)$ would be:

$$\phi_1(v) = \frac{2}{6}(v(1) - v(\emptyset) + \frac{1}{6}(v(12) - v(2)) + \frac{1}{6}(v(13) - v(3)) + \frac{2}{6}(v(123) - v(23))$$

Similarly,

$$\phi_2(v) = \frac{2}{6}(v(2) - v(\emptyset) + \frac{1}{6}(v(12) - v(1)) + \frac{1}{6}(v(23) - v(3)) + \frac{2}{6}(v(123) - v(13))$$

$$\phi_3(v) = \frac{2}{6}(v(3) - v(\emptyset) + \frac{1}{6}(v(13) - v(1)) + \frac{1}{6}(v(23) - v(2)) + \frac{2}{6}(v(123) - v(12))$$

It can be easily seen that

$$\phi_1(v) = 200; \ \ \phi_2(v) = 50; \ \ \phi_3(v) = 50$$

Also, it can be easily verified from the above expressions that

$$\phi_1(v) + \ \ \phi_2(v) + \ \ \phi_3(v) = v(123)$$

It can also be verified easily that $\phi = (100, 100, 100)$ for Version 1 and Version 4 of the game while $\phi = (150, 150, 0)$ for Version 2 of the game.

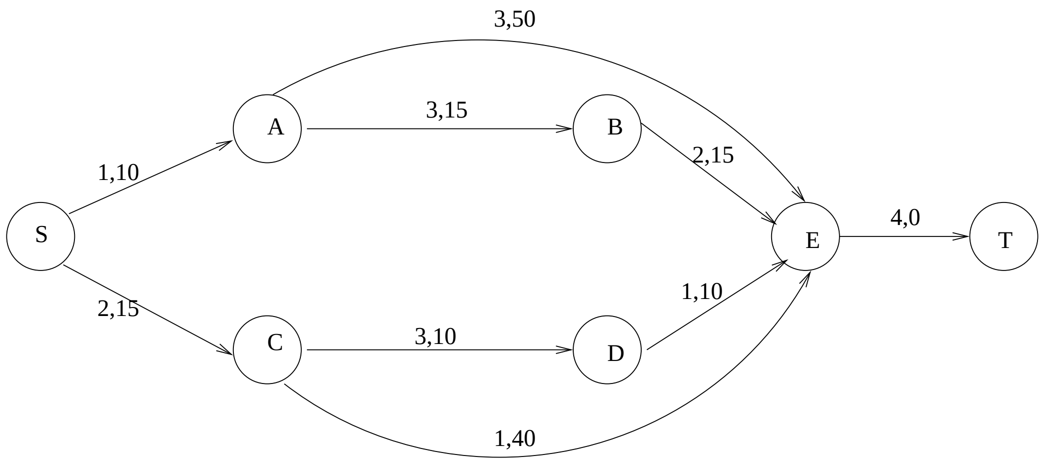

Figure 2: A logistics network

2.2 Example 2: Glove Market

Consider the left-glove - right-glove game with 2,000,001 players, having one extra right glove supplier. We have seen that the core of this game is empty. Here, the Shapley value of any right-glove supplier is the probability that he would find in any coalition more left glove suppliers than right-glove suppliers. Here the Shapley value for each right-glove supplier is 0.499557 and for each left glove supplier is 0.500443. Note that the Shapley value is able to capture the effect of coalitional forces better than the core.

2.3 Example 3: Apex Game

In this game, there are five players. Player 1 is called the big player and the other players are called small players. The big player with one or more small players can earn a worth of 1. The four small players together can also earn 1.

$$\begin{aligned} N &= \{1, 2, 3, 4, 5\} \\ v(C) &= 1 \text{ if } 1 \in C \text{ and } |C| \geq 2 \\ &= 1 \text{ if } |C| \geq 4 \\ &= 0 \text{ otherwise} \end{aligned}$$

The core of this game is empty. The Shapley value is

$$\phi(v) = (\frac{3}{5}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10})$$

2.4 Example 4: A Logistics Game

Figure 1 shows a logistics network that provides connectivity between two important cities S and T. There are five hubs A, B, C, D, E which are intermediate points from S to T. The transportation is provided by service providers 1, 2, 3, 4. We have already seen this example in Chapter 30.

We have formulated this as a TU game with $N = \{1, 2, 3, 4\}$ and with characteristic function

$$v(1) = v(2) = v(3) = v(4) = 0$$

$$v(12) = v(13) = v(14) = v(23) = v(24) = v(34) = v(234) = v(123) = 0$$

$$v(134) = 100 - 60 = 40; \;\; v(124) = 100 - 55 = 45; \;\; v(1234) = 100 - 35 = 65$$

It can be shown that the Shapley values here are given by

$$\phi(v) = (20, 20, 5, 20$$

This can be shown to belong to the core of this game.

3 Proof of the Shapley Theorem

The proof proceeds in two parts.

- Part 1: To show that the formula for $\phi_i(v)$ satisfies all the three axioms.
- Part 2: To show that there exists exactly one mapping ϕ that satisfies all the three axioms.

3.1 Proof of Part 1

Showing Symmetry

Observe that in the formula for $\phi_i(v)$, what only matters about a coalition is whether it contains i and the number of players it contains. Thus relabeling does not affect the value in any way. This observation clearly shows that symmetry (Axiom 1) is satisfied.

Showing Linearity

To show linearity, we have to show that

$$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w)$$

We have, by definition, for any $p \in [0, 1]$,

$$(pv + (1-p)w)(C) = pv(C) + (1-p)w)(C) \; \forall C \subseteq N \tag{1}$$

Note that

$$\phi_i(pv + (1-p)w) = \sum_{C \subseteq N-i} \frac{|C|! \;\, (n - |C| - 1)!}{n!} \; \big((pv + (1-p)w))(C \cup \{i\}) - (pv + (1-p)w)(C)\big)$$

By expanding this and applying (1), linearity can be established.

Showing Carrier Axiom

Suppose D is a carrier of (N, v). Then, we know that

$$v(C) = v(C \cap D)\ \forall C \subseteq N$$

We have also seen that

$$v(\{i\}) = 0 \quad \forall \ i \in N \setminus D \quad \text{and} \quad v(D) = v(N)$$

If we take a look at the formula for the Shapley value, it is very clear that

$$\phi_i(v) = 0\ \forall i \in N \setminus D$$

since

$$v(C \cup \{i\}) = v((C \cup \{i\}) \cap D) = v(C \cap D) = v(C)$$

We have to show for all carriers D that

$$\sum_{i \in D} \phi_i(v) = v(D)$$

Substituting the formula for $\phi_i(v)$ from the Shapley theorem and simplifying, we can show that

$$\sum_{i \in N} \phi_i(v) = v(N)$$

Since D is a carrier, we have $v(D) = v(N)$, hence the carrier axiom follows.

3.2 Proof of Part 2

Here we show that there exists exactly one mapping ϕ that satisfies the three axioms. First we prove that the mapping ϕ is a linear transformation by making the following observations.

- Let (N, z) be the coalitional game that assigns worth zero to every coalition, that is, $z(C) = 0\ \ \forall C \subseteq N$. Then Axiom 3 (carrier axiom) implies that

$$\phi_i(z) = 0\ \forall i \in N \tag{2}$$

- From Axiom 2, we have

$$\phi_i(pv + (1-p)w) = p\phi_i(v) + (1-p)\phi_i(w)$$

  Choosing $w = z$ in the above, we get

$$\phi_i(pv) = p\phi_i(v) \ \ \forall i \in N \tag{3}$$

Equations (2) and (3) together with the linearity axiom imply that ϕ is a linear transformation.

Suppose $L(N)$ denotes the set of all non-empty subsets of N. Clearly, $|L(N)| = 2^n - 1$ and hence $\mathbb{R}^{|L(N)|}$ is a (2^n-1)-dimensional vector space. Let $D \subseteq N$ be any coalition. Define for D, a coalitional game (N, w_D):

$$\begin{aligned} w_D(C) &= 1 \quad \text{if } D \subseteq C \\ &= 0 \quad \text{otherwise} \end{aligned}$$

This implies that a coalition C has worth 1 in w_D if it contains all the players in D and has worth zero otherwise. The game (N, w_D) is called the simple D-carrier game. To get a feel for this game, we discuss a simple example before resuming the proof.

Let $N = \{1, 2, 3\}$. Let $w_1, w_2, w_3, w_{12}, w_{13}, w_{23}, w_{123}$ be the characteristic functions of the D-carrier games corresponding to the coalitions $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$, respectively. For instance, w_{12} would be:

$$\begin{aligned}
w_{12}(1) &= 0 \\
w_{12}(2) &= 0 \\
w_{12}(3) &= 0 \\
w_{12}(12) &= 1 \\
w_{12}(13) &= 0 \\
w_{12}(23) &= 0 \\
w_{12}(123) &= 1
\end{aligned}$$

We can write the seven different w values as follows. Note that each one is a 7-tuple.

$$\begin{aligned}
w_1 &= (1, 0, 0, 1, 1, 0, 1) \\
w_2 &= (0, 1, 0, 1, 0, 1, 1) \\
w_3 &= (0, 0, 1, 0, 1, 1, 1) \\
w_{12} &= (0, 0, 0, 1, 0, 0, 1) \\
w_{13} &= (0, 0, 0, 0, 1, 0, 1) \\
w_{23} &= (0, 0, 0, 0, 0, 1, 1) \\
w_{123} &= (0, 0, 0, 0, 0, 0, 1)
\end{aligned}$$

We will be showing subsequently that these seven vectors is a linearly independent set that spans $\mathbb{R}^7$.

Continuing the proof of Part 2, note that D is a carrier of (N, w_D) since

$$w_D(C) = w_D(C \cap D) \;\; \forall C \subseteq N$$

In fact, $D \cup \{j\}$ is also a carrier $\forall j \in N$. Also

$$\begin{aligned}
w_D(N) &= 1 \quad \text{since } D \subseteq N \\
w_D(D) &= 1 \quad \text{since } D \subseteq D
\end{aligned}$$

Application of the carrier axiom leads to the following

$$\begin{aligned}
\sum_{i \in D} \phi_i(w_D) &= 1 = w_D(D) = w_D(N) \;\; \text{and} \\
\phi_j(w_D) &= 0 \;\; \forall j \notin D
\end{aligned}$$

The above holds because both D and $D \cup \{j\}$ are carriers of w_D.

Now, by the symmetry axiom, all players in D should get the same payoff. This is because the players in D all contribute equally to the characteristic function w_D. There is no way to distinguish them in any manner, so relabeling can be freely done. This implies:

$$\phi_i(w_D) = \frac{1}{|D|} \ \ \forall i \in D$$
$$\phi_j(w_D) = 0 \ \ \forall j \notin D$$

For each coalition $D \subseteq N$, there is a game (N, w_D) and therefore we have $2^n - 1$ such games. Each game is represented by a characteristic function which is a vector of $2^n - 1$ elements. Thus we have $2^n - 1$ vectors each of size $2^n - 1$. We now show that these are linearly independent in the space $\mathbb{R}^{2^n-1}$. To show linear independence, we have to show that

$$\sum_{D \in L(N)} \lambda_D w_D = (0, 0, \ldots, 0) \Rightarrow \lambda_D = 0 \ \ \forall D \in L(N)$$

Suppose not. Now,

$$\sum_{D \in L(N)} \lambda_D w_D = (0, 0, \ldots, 0) \ \ \Rightarrow \ \ \sum_{D \in L(N)} \lambda_D w_D(C) = 0 \ \ \forall C \subseteq N$$

Let $C \subseteq N$ be any coalition of minimal size such that $\lambda_C \neq 0$. Based on the observation that D is either a subset of C or not, we can write:

$$\sum_{D \subseteq C} \lambda_D w_D(C) + \sum_{\neg(D \subseteq C)} \lambda_D w_D(C) = 0 \Rightarrow \ \ \sum_{D \in L(N)} \lambda_D w_D(C) = 0 \ \ \forall C \subseteq N$$

We know that $w_D(C) = 1 \ \ \forall D \subseteq C$ and $w_D(C) = 0$ otherwise (that is, if D is not a subset of C). Therefore, the above implication leads to

$$\sum_{D \subseteq C} \lambda_D = 0$$

Since C is a subset of minimal size for which $\lambda_C \neq 0$, we have that $\lambda_D = 0$ for all subsets $D \subset C$. Thus the above sum is equal to λ_C. This means $\lambda_C = 0$. This is a contradiction ! Thus the set $\{w_D : D \in L(N)\}$ is a basis of the space $\mathbb{R}^{2^n-1}$. Since a linear transformation is completely determined by what it does on a basis of the vector space, the mapping ϕ is unique.

4 Alternative Formulae for Shapley Value

We now describe several alternative ways of developing the Shapley value. Each of these will provide an interesting interpretation for this solution concept.

4.1 Approach 1

The Shapley value can be written is :

$$\phi_i(v) = \sum_{C \subseteq N-i} \frac{|C|!(n-|C|-1)!}{n!} \{v(N \setminus C) - v(C)\}$$

This is because the coefficient of $\{v(N \setminus C) - v(C)\}$ is the same as that of $\{v(C \cup \{i\}) - v(C)$. The above expression shows that the value of each player depends only on the difference between the worths of the complementary coalitions. For each pair of complementary coalitions C and $N \setminus C$, the values of players in $N \setminus C$ increase as $v(N \setminus C) - v(C)$ increase while the values of players in C increase as $v(C) - v(N \setminus C)$ increases. This result has implications for the analysis of cooperative games with transferable utility in strategic form.

4.2 Approach 2

Another formula for Shapley value is:

$$\phi_i(N, v) = \sum_{\substack{C \subseteq N \\ i \in C}} \frac{(|C|-1)!(n-|C|)!}{n!} \{v(N) - v(N \backslash \{i\})\}$$

It is to be noted that

$$\frac{(|C|-1)!(n-|C|)!}{n!}$$

is the probability that in a random permutation, the coalition C arises as the union of $\{i\}$ with all the predecessors of i.

4.3 Approach 3

Let $C \subseteq N$ and $i \notin C$. Let $m(C, i)$ denote the marginal contribution of i to C. Then

$$m(C, i) = v(C \cup \{i\}) - v(C)$$

Let π be any permutation of $\{1, \ldots, n\}$. Let $P(\pi, i)$ represent the set of players who are predecessors of i in the permutation π. That is,

$$P(\pi, i) = \{j : \pi(j) < \pi(i)\}$$

We have seen that $\phi_i(N, v)$ is the average marginal contribution of i to any coalition of N assuming all orderings are equally likely. Another way of interpreting this is that $\phi_i(N, v)$ is the average marginal contribution of i to the set of his predecessors where the average taken over all permutations equally likely. We therefore get

$$\phi_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi} m(P(\pi, i), i) \tag{4}$$

where Π is the set of all permutations of $N = \{1, \ldots, n\}$. Similarly if

$$S(\pi, i) = \{j : \pi(i) < \pi(j)\}$$

denotes the set of all players who are successors of i in the permutation π, we also get

$$\phi_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi} m(S(\pi, i), i) \tag{5}$$

One can verify (4) and (5) independently satisfy the equations (2) and (3) that define the Shapley value.

5 Convex Games

A TU game (N, v) is said to be convex if for every $i \in N$, the marginal contribution of player i to larger coalitions is larger. That is:

$$v(D \cup \{i\}) - v(D) \geq v(C \cup \{i\}) - v(C) \;\; \forall C \subseteq D \subseteq N \text{ and } \; \forall i \notin D$$

Another way of writing this would be:

$$v(D \cup \{i\}) - v(C \cup \{i\}) \geq v(D) - v(C) \;\; \forall C \subseteq D \subseteq N \text{ and } \; \forall i \notin D$$

We now recall the definition of two other classes of coalitional games.

Supermodular Games

A TU game (N, v) is said to be supermodular if

$$v(C \cup D) \geq v(C) + v(D) - v(C \cap D) \ \ \forall \ \ C, D \subseteq N$$

It can be shown (see Exercise) that a supermodular game is nothing but a convex game.

Superadditive Games

Recall that a TU game (N, v) is said to be superadditive if

$$v(C \cup D) \geq v(C) + v(D) \ \ \forall \ \ C, D \subseteq N$$

Clearly, a superadditive game is supermodular and hence convex.

5.1 Some Important Results

We summarize below some of the important properties that convex games satisfy.

- The core of a convex game is always non-empty. Note in general that the core of a coalitional game could be empty which means there is no stable allocation that is sustainable..
- The Shapley value of a convex game always belongs to the core of the game. Note in general that the Shapley value need not be a member of the core even if the core is non-empty. This means that the Shapley may not always provide a stable allocation. On the other hand, it represents a *fair* allocation in the sense of satisfying the three axioms of Shapley.
- The Shapley value of a convex game can be shown to be the center of gravity of the core of the game. This fact can be used to design efficient algorithms for computing the Shapley value of a convex game.

We now prove the result that the Shapley value of a convex game always belongs to the core of the game.

5.2 Shapley Value of a Convex Game Belongs to the Core

To prove this important result, we first show the following. Let $C \subseteq N, \ \ D = C \cup \{i\}$ and let $i \in C$. Then

$$\phi_i(C, v) \leq \phi_i(D, v)$$

To show this, consider a permutation π of C. Let $m(\pi, i)$ be the marginal contribution of i to its predecessors in C. Then

$$\phi_i(C, v) = \frac{1}{|C|!} \sum_{\pi \in \Pi(C)} m(\pi, i)$$

where $\Pi(C)$ denotes the set of all permutations of elements of C. Recall that $D = C \cup \{i\}$. Let $m'(\pi, i)$ be the average marginal contribution of i to its predecessors in D when the average is taken over the $|D|!$ permutations of D differing from the permutation π of C only by the placement of i. Then

$$\phi_i(D, v) = \frac{1}{|C|!} \sum_{\pi \in \Pi(C)} m'(\pi, i)$$

It is clear that $\forall \pi \in \Pi(C)$, we have

$$m'(\pi, i) \geq m(\pi, i)$$

because

- if we place j after i, then the marginal contribution of i to its predecessors in D is still $m(\pi, i)$.
- if we place j before i, then due to convexity,

$$m'(\pi, i) \geq m(\pi, i)$$

Now such a property will be satisfied in general for any set $D \supseteq C$. Thus we have shown that

$$i \in C \subseteq D \Rightarrow \phi_i(C, v) \leq \phi_i(D, v)$$

Thus

$$\sum_{i \in C} \phi_i(C, v) \leq \sum_{i \in C} \phi_i(D, v)$$

That is

$$v(C) \leq \sum_{i \in C} \phi_i(D, v)$$

Consider the allocation

$$(\phi_i(D, v))_{i \in D}$$

It is clear that no coalition of D can block this allocation. Consider the allocation

$$(\phi_i(N, v), \ldots, \phi_n(N, v))$$

It is clear that no coalition can block this and moreover

$$\phi_i(N, v) + \ldots + \phi_n(N, v)) = v(N)$$

Thus the Shapley value allocation belongs to the core.

6 To Probe Further

The material discussed in this chapter draws upon mainly from the the book by Myerson [2]. The volume edited by Roth [3] embodies a comprehensive account of Shapley value and extensions until 1988. Computation of Shapley value is a hard problem and clearly it has exponential time complexity. The book by Chalkiadakis, Elkind, and Wooldridge [4] has a detailed discussion on this important topic.

7 Problems

1. Show using the expression for Shapley value that the sum of Shapley values of all players will be equal to the value of the grand coalition.

2. (Straffin 1993) [5]. Consider the following variant of the real estate example. Player 1 has a value of Rs. 1 million; player 2 has value of Rs. 2 million; and player 3 has a value of Rs. 3 million for the house. Player 2 has Rs. 3 million cash, so also player 3. Formulate an appropriate TU game and compute the Shapley value.

3. (Straffin 1993) [5]. Consider a three person superadditive game with $v(1) = v(2) = v(3) = 0; v(12) = a; v(13) = b; v(23) = c; v(123) = d$ where $0 \leq a, b, c \leq d$. Compute the Shapley value for this game.

4. Find the Shapley value of the *communication satellites game* [5] defined as follows:

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = 5.2; \quad v(13) = 2.5; \quad v(23) = 3; \quad v(123) = 5.2$$

5. Consider a game with five players where player 1 is called a big player and the others are called small players. The big player with one or more small players can earn a worth of 1. The four small players together can also earn 1. Let

$$\begin{aligned} N &= \{1, 2, 3, 4, 5\} && (6) \\ v(C) &= 1 \text{ if } 1 \in C \text{ and } |C| \geq 2 && (7) \\ &= 1 \text{ if } |C| \geq 4 && (8) \\ &= 0 \text{ otherwise} && (9) \end{aligned}$$

   Compute the Shapley value for this game.

6. Let us consider a version of divide the dollar problem with 4 players and total worth equal to 400. Suppose that any coalition with three or more players will be able to achieve the total worth. Also, a coalition with two players will be able to achieve the total worth only if player 1 is a part of the two player coalition. Set up a characteristic function for this TU game and compute the Shapley value.

7. There are four players $\{1, 2, 3, 4\}$ who are interested in a wealth of 400 (real number). Any coalition containing at least two players and having player 1 would be able to achieve the total wealth of 400. Similarly, any coalition containing at least three players and containing player 2 also would be able to achieve the total wealth of 400. Set up a characteristic form game for this situation and compute the Shapley value.

8. Give simple examples of 2 player situations where

   - The core is empty and Nash Bargaining Solution (NBS) = Shapley Value (SV)
   - The core is empty and NBS is not the same as SV
   - The core is non-empty and NBS = SV
   - The core is non-empty and NBS is not the same as SV

9. Give simple examples of TU games where

   - The core is non-empty and the Shapley value belongs to the core

- The core is non-empty and the Shapley value does not belong to the core

10. Consider the following characteristic form game with three players. $v(1) = v(2) = v(3) = 0; v(12) = a; v(13) = b; v(23) = c; v(123) = 1$. Assume that $0 \leq a, b, c \leq 1$.
    (a) Find the conditions under which the core is non-empty.
    (b) Compute the Shapley value.
    (c) Assuming the core is non-empty, does the Shapley value belong to the core? Under what conditions will the Shapley value belong to the core of this game.

11. Consider the following situation. It is required to lease a certain quantity of telecom bandwidth from Bangalore to Delhi. There are two service providers in the game $\{1, 2\}$. Provider 1 offers a direct service from Bangalore to New Delhi with a bid of 100 million rupees. Provider 1 also offers a service from Mumbai to Delhi with a bid of 30 million rupees. On the other hand, Provider 2 offers a a direct service between Bangalore and Delhi with a bid of 120 million dollars and a service from Bangalore to Mumbai with a bid of 50 million rupees. Set up a TU coalitional game for this situation. Compute the core and Shapley value for this game.

12. Show that a convex game is always supermodular. That is,

$$v(C \cup D) \geq v(C) + v(D) - v(C \cap D) \ \ \forall \ \ C, D \subseteq N$$

8 Appendix: An Alternative Development of the Shapley Value

We start with a two player TU game (N, v), with $N = \{1, 2\}$. The gains from cooperation is given by

$$v(N) - v(1) - v(2)$$

Let us say we are interested in coming up with an egalitarian allocation, which means the gains from cooperation is equally distributed between the two players. Let

$$\phi(N, v) = (\phi_1(N, v), \phi_2(N, v))$$

be such an egalitarian allocation. It is easy to see that

$$\begin{aligned} \phi_1(N, v) &= v(1) + \frac{1}{2}(v(N) - v(1) - v(2)) \\ \phi_2(N, v) &= v(2) + \frac{1}{2}(v(N) - v(1) - v(2)) \end{aligned}$$

The above equations may be rewritten as

$$\begin{aligned} \phi_1(N, v) - v(1) &= \phi_2(N, v) - v(2) \\ \phi_1(N, v) + \phi_2(N, v) &= v(N) \end{aligned}$$

It makes sense to denote $v_i = \phi_i(\{i\}, v)$. The above two equations can now be written as

$$\begin{aligned} \phi_1(N, v) - \phi_1(\{1\}, v) &= \phi_2(N, v) - \phi_2(\{2\}, v) \\ \phi_1(N, v) + \phi_2(N, v) &= v(N) \end{aligned}$$

The first equation above says that the utility difference (utility obtained with cooperation minus utility without cooperation) for player 1 is the same as that for player 2. The second equation says that the sum of the utilities is equal to the worth of the grand coalition. Another way of describing the first equation is: what player 1 gets out of the presence of player 2 is the same as what player 2 gets out of the presence of player 1. See Figure 2.

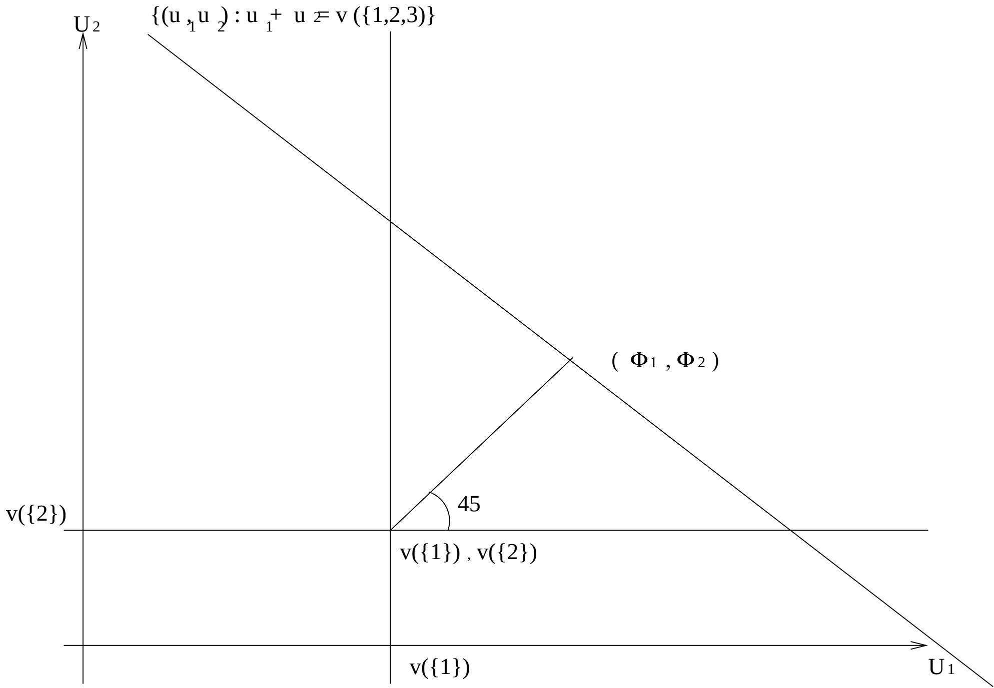

Figure 3: Alternative representation

We have earlier seen that the Nash bargaining solution is the solution which is both λ-egalitarian and λ-utilitarian with λ equal to the natural scale factor. Thus for a two person game, the Shapley value could be different from the Nash bargaining solution. Moreover note that the Nash bargaining solution depends on how we choose (v_1, v_2). The above egalitarian perspective can be generalized to any TU game (N, v) as follows. Given any coalition $C \subseteq N$, let (C, v) denote the TU game obtained by restricting $v(.)$ to the subsets of C. This is called the subgame (C, v). An egalitarian solution for the game (N, v) is a family of numbers $\{\phi_i(C, v)\}_{i \in C} C \subseteq N$ such that for every subgame (C, v) and players $i, j \in C$, the utility differences are preserved in the following way:

$$\phi_i(C, v) - \phi_i(C \backslash \{i\}, v) = \phi_j(C, v) - \phi_j(C \backslash \{i\}, v) \;\; \forall C \subseteq N \;\; \forall i, j \in C \quad (10)$$

$$\sum_{i \in C} \phi_i(C, v) = v(C) \;\; \forall C \subseteq N \quad (11)$$

An alternative definition for the Shapley value is: The Shapley value of (N, v), denoted by $\phi(N, v) = (\phi_1(N, v), \ldots, \phi_n(N, v))$ is the unique outcome that satisfies (2) and (3).

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