Game Theory

Lecture Notes By

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COOPERATIVE GAME THEORY
Other Solution Concepts

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

Thus far, we have studied two key solution concepts (The core and the Shapley value) in cooperative games. A rich variety of solution concepts have been studied in cooperative game theory. In this chapter, we briefly study five other concepts: (1) Stable Sets; (2) Bargaining Sets; (3) Kernel; (4) Nucleolus; and (5) Gately Point.

1 Stable Sets

We first present several definitions before introducing the concept of stable sets.

Excess

Given a TU game (N, v), a coalition C, and an allocation $x = (x_1, \ldots, x_n)$, the excess of C at x is defined as

$$e(C, x) = v(C) - \sum_{i \in C} x_i$$

The excess $e(C, x)$ is the *net transferable utility* that coalition C would be left with after allocating x_i to player i where $i \in C$. If $e(C, x) \geq 0$, the implication is that the coalition, on its own, would be able to achieve its share of the allocation x.

Example: Version 3 of Divide the Dollar

Recall this game:

$$\begin{aligned} N &= \{1, 2, 3\} \\ v(1) &= v(2) = v(3) = v(23) = 0 \\ v(12) &= v(13) = v(123) = 300 \end{aligned}$$

Consider allocation $x = (100, 100, 100)$. Then we have

$$\begin{aligned} e(1,x) &= -100 \\ e(2,x) &= -100 \\ e(3,x) &= -100 \\ e(23,x) &= -200 \\ e(12,x) &= 100 \\ e(13,x) &= 100 \\ e(123,x) &= 0 \end{aligned}$$

Suppose $y = (200, 50, 50)$ then

$$\begin{aligned} e(1,y) &= -200 \\ e(2,y) &= e(3,y) = -50 \\ e(23,y) &= -250 \\ e(13,y) &= e(12,y) = 50 \\ e(123,y) &= 0 \end{aligned}$$

If $z = (300, 0, 0)$, we have

$$\begin{aligned} e(1,z) &= e(2,z) = e(3,z) = 0 \\ e(23,z) &= e(12,z) = e(13,z) = 0 \\ e(123,z) &= 0 \end{aligned}$$

Domination

An imputation $x = (x_1, \ldots, x_n)$ is said to dominate another imputation $y = (y_1, \ldots, y_n)$ if there is some coalition $C \subseteq N$ such that

1. $e(c, x) \geq 0$
2. $x_i > y_i \;\; \forall i \in C$

An imputation $x = (x_1, \ldots, x_n)$ is said to be *undominated* if no other imputation dominates it. One can immediately note that the the core of a TU game is simply the collection of all undominated imputations.

Internal Stability

Given a TU game (N, v), a set of imputations Z is said to be *internally stable* if

$$\forall x, y \in Z, \; \forall C \subseteq N, \;\; x_i > y_i \; \forall i \in C \Rightarrow e(C, x) < 0$$

This means no imputation in Z is dominated by any other imputation in Z. Internal stability implies that if an imputation y belonging to Z is proposed, then no coalition C can block this since a strictly better, feasible outcome does not exist in Z for the players in C.

External Stability

Given a TU game (N, v), a set of imputations Z is said to be *externally stable* if every imputation not in Z is dominated by some imputation in Z. External stability can be described as follows: For all imputations y of (N, v), $y \notin Z \Rightarrow$ there exists an $x \in Z$ such that there is at least one coalition $C \subseteq N$ for which

$$x_i > y_i \; \forall i \in C \; \Rightarrow e(C, x) \geq 0$$

External stability implies that if an imputation y not belonging to Z is proposed, then there is at least one coalition C of players that could block the adoption of y by insisting on getting their share of some strictly better outcome in Z that is feasible for them.

Stable Sets

This is a solution concept that was first studied by von Neumann and Morgenstern in 1944. A *stable set* is also called a Von-Neumann-Morgenstern solution. A stable set of a TU game(N, v) is a set of imputations Z satisfying internal stability and external stability.

Example: Majority Voting Game

Recall this game:

$$\begin{aligned} N &= \{1, 2, 3\} \\ v(1) &= v(2) = v(3) = v(23) = 0 \\ v(12) &= v(23) = v(123) = 300 \end{aligned}$$

One possible stable set here is $Z = \{(150, 150, 0), (150, 0, 150), (0, 150, 150)\}$. The set of imputations for this example is given by

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0; x_2 \geq 0; x_3 \geq 0; x_1 + x_2 + x_3 = 300\}$$

To get a feel for the definition of a stable set, let us try out different test cases.

- Suppose the imputation $(300, 0, 0) \notin Z$ is suggested. Now, the imputation $(0, 150) \in Z$ is such that the coalition $\{2, 3\}$ will block $(300, 0, 0)$.
- Let us say we explore the imputation $x = (100, 100, 100) \notin Z$
  - The imputation $(150, 150, 0) \in Z$ is such that the coalition $\{1, 2\}$ will block x.
  - The imputation $(150, 0, 150) \in Z$ is such that the coalition $\{1, 3\}$ will block x.
  - The imputation $(0, 150, 150) \in Z$ is such that the coalition $\{2, 3\}$ will block x.

  Thus all imputations in Z dominate x
- On the other hand, we observe that $(150, 150, 0) \in Z$ is not dominated by either $(150, 0, 150)$ or $(0, 150, 150)$.

- For the above example, three other stable sets are given by

$$(0 \leq y_1 \leq 150, 0 \leq y_2 \leq 150, 0 \leq y_3 \leq 150) : \{x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = y_1; x_2 \geq 0; x_3 \geq 0; x_1+x_2+x_3 = 300\}$$
$$\{x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = y_2; x_1 \geq 0; x_3 \geq 0; x_1 + x_2 + x_3 = 300\}$$
$$\{x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = y_3; x_1 \geq 0; x_2 \geq 0; x_1 + x_2 + x_3 = 300\}$$

Given a TU game, a stable set may or may not exist. In fact, the existence of a stable set was an open problem for a long time until William Lucas finally constructed, in 1968, a 10 person game for which there is no stable set. This was such a messy problem that there was great relief for researchers in this area when this counter example was found. Stable sets can offer valuable insights into coalitional dynamics, however the limitations arise because

- there may be many stable sets
- there may not be any stable set
- there may have a quite complex structure.

2 Bargaining Set

This solution concept was introduced by Aumann and Maschler in 1964. The intuition behind this solution concept is that a player would be deterred from objecting to a proposed payoff allocation because of the apprehension that the objection might prompt a counter objection by another player. We shall first formalize the notion of an objection and a counter objection.

Objection

An objection by a player i against another player j and a pay off allocation x is a pair (y, C) where y is another pay off allocation and C is a coalition such that

$$\begin{aligned} i &\in C \\ j &\notin C \\ e(C, y) &= 0 \\ y_k &= x_k \;\; \forall k \in C \end{aligned}$$

Example of an Objection: Majority Voting Game

Recall again the three person majority voting game:

$$\begin{aligned} N &= \{1, 2, 3\} \\ v(1) &= v(2) = v(3) = 0 \\ v(12) &= v(23) = v(13) = v(123) = 300 \end{aligned}$$

Let $i = 1, j = 2$, and $x = (50, 100, 150)$. An objection by player 1 against player 2 and the payoff allocation x is a pair (y, C) where

$$\begin{aligned} y &= (125, 0, 175) \\ C &= \{1, 3\} \end{aligned}$$

Note that $i \in C, j \notin C, e(c, y) = 0; y_1 > x_1$ and $y_3 > x_3$.

Counterobjection

Given a player i's objection (y, C) against player j and a payoff allocation x, a counterobjection by player j is any pair (z, D) where z is another payoff allocation and D is a coalition such that

$$\begin{aligned} i &\in D \\ i &\notin D \\ C &\cap D \neq \phi \\ e(D, z) &= 0 \\ z_k &\geq x_k \ \forall k \in D \\ z_k &\geq y_k \ \forall k \in C \cap D \end{aligned}$$

Example of a Counterobjection: Majority Voting Game

For the objection described in the previous example, a counter objection by player 2 would be the pair (z, D) where

$$\begin{aligned} z &= (0, 125, 175) \\ D &= \{2, 3\} \\ C \cap D &= \{3\} \\ e(D, z) &= 0 \\ z_2 &\geq x_2; \ \ z_3 \geq x_3; \ \ z_3 \geq y_3 \end{aligned}$$

Before we define a bargaining set, we introduce some additional notation. Let (N, v) be a TU game and suppose Q is a partition of (N, v). We define

$$I(Q) = \{x \in \mathbb{R}^n : x_i \geq v(\{i\}) \ \forall i \in N, \ \sum_{i \in C} x_i = v(C) \ \ \forall C \in Q\}$$

One can immediately note that if $Q = \{N\}$, that is the partition consisting of only one part namely the entire set N, then $I(Q)$ is exactly the set of all imputations of (N, v).

Definition: Given a TU game (N, v) and a partition Q of N, a bargaining set is a collection of payoff allocations $x \in \mathbb{R}^n$ such that

1. $x \in I(Q)$
2. For any coalition D in Q and for any two players $i, j, \in D$, there exists a counterobjection to any objection by i against j and x.

Example: Apex Game

Recall the apex game that we have studied earlier:

$$\begin{aligned} N &= \{1, 2, 3, 4, 5\} \\ v(C) &= 1 \text{ if } 1 \in C \text{ and } |C| \geq 2 \\ &= 1 \text{ if } |C| \geq 4 \\ &= 0 \text{ otherwise} \end{aligned}$$

In the above game, player 1 is called the big player and the other players are called small players. We have seen that the core and the Shapley value are given by

$$\begin{aligned}\text{Core }(N, v) &= \phi \\ \Phi(N, v) &= \left(\frac{6}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)\end{aligned}$$

Consider the partition $Q = \{N\}$. With respect to this partition, the bargaining set can be shown to be

$$\left\{(1 - 4\lambda, \lambda, \lambda, \lambda, \lambda) : \frac{1}{13} \leq \lambda \leq \frac{1}{7}\right\}$$

An immediate observation we make is that the Shapley value belongs to this bargaining set. There are three representative situations here that we need to look at so as to confirm that the above is the bargaining set.

- *Situation 1:* Assume that the small players do not all get the same payoff. In this case, suppose small player i gets strictly less than small player j. Then player i would have an objection in collaboration with the big player 1 that player j would not be able to counter.

- *Situation 2:* Suppose all the small players manage to get a payoff greater than $\frac{1}{7}$. Then player 1 would have, for example, an objection $\left(\left(\frac{3}{7}, \frac{4}{7}, 0, 0, 0\right), \{1, 2\}\right)$ for which player 3 or player 4 or player 5 would not be able to counter.

- *Situation 3:* If all the small players get an amount less than $\frac{1}{13}$, then player 1 would get greater than $\frac{9}{13}$. Then each of the small players can immediately object. For example, player 2 could have an objection $\left(\left(0, \frac{1}{13}, \frac{4}{13}, \frac{4}{13}, \frac{4}{13}\right), \{2, 3, 4, 5\}\right)$ which player 1 would not be able to counter.

For the above apex game, if $Q = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \}$ then the bargaining set would be

$$\left\{(1 - \alpha, \alpha, 0, 0, 0) : \frac{1}{4} \leq \alpha \leq \frac{1}{2}\right\}$$

Example

This example is taken from [1]. Consider a three player game with $N = \{1, 2, 3\}$ and characteristic function:

$$v(1) = v(2) = v(3) = 0$$
$$v(12) = 60;\ v(13) = 80;\ v(23) = 100$$
$$v(123) = 105$$

It can be shown that the core is empty here. The Shapley values can be shown to be

$$\phi_1(N, v) = 25;\ \phi_2(N, v) = 35;\ \phi_3(N, v) = 45$$

Consider the situation facing players 2 and 3 if they decide to form a coalition, leaving out player 1. the two players 2 and 3 will have to share 100 units and if they do admit player 1 into their coalition, they will have to share 105 units among three players. In the latter case, players 2 and 3 may actually end up getting less than what they would have possibly got if only two of them formed a coalition.

In addition to this, there is also time and effort spent in extra negotiation, so players 2 and 3 may desist from inviting player 1 to join their coalition. This leads to the partition $\{\{1\},\{2,3\}\}$. It can be shown that $(0,40,60)$ is the only stable allocation for this partition. In fact, for each of the five possible coalition structures, there is exactly one bargaining set in this case as shown in Table 1.

Coalition structure (partition)	Bargaining Set
$\{\{1\},\{2\},\{3\}\}$	$\{(0,0,0)\}$
$\{\{1,2\},\{3\}\}$	$\{(20,40,0)\}$
$\{\{1,3\},\{2\}\}$	$\{(20,0,60)\}$
$\{\{1\},\{2,3\}\}$	$\{(0,40,60)\}$
$\{\{1,2,3\}\}$	$\{(15,35,55)\}$

Table 1: Bargaining sets for different partitions

If the value of the grand coalition changes to 135 instead of 105, then the bargaining set for the partition $\{\{1,2,3\}\}$ will be exactly equal to the core:

$$\{(x_1,x_2,x_3) \in \mathbb{R}^3 : \ x_1 \geq 0; \ x_2 \geq 0; \ x_3 \geq 0; x_1 \leq 35; \ x_2 \leq 55; x_3 \leq 55; \ x_1+x_2+x_3 = 135\}$$

The bargaining sets for the other four coalition structures will remain the same as before.

Some Observations on the Bargaining Set

- The core is a (possibly empty) subset of the bargaining set of (N,v) relative to the partition $Q = \{N\}$.
- Given a partition Q, if $I(Q)$ is non-empty, then the bargaining set relative to Q is non-empty.
- Recall that a TU game (N,v) is said to be superadditive if for all coalitions $C, D \subseteq N$,

  $$C \cap D = \Phi \Rightarrow v(C \cup D) \geq v(C) + v(D)$$

  If the game (N,v) is superadditive, then for any partition Q, the set $I(Q)$ is non-empty and hence the bargaining set with respect to Q is also non-empty.
- the Shapley value need not belong to the bargaining set corresponding to any given partition. Thus the allocations suggested by a bargaining set need not be fair in the sense of Shapley value.

3 Kernel

This solution concept was proposed by Davis and Maschler [2]. The Kernel of a TU game (N,v) is defined with respect to a partition Q of N (like the bargaining set). In fact, like the bargaining set, the Kernel is also a subset of $I(Q)$. The intuition behind the Kernel is that if two players i and j belong to the same coalition in Q, then the highest excess that i can make in a coalition without j should be the same as the highest excess that j can make in a coalition without i. While defining the Kernel, we focus on the maximum excess a player would be able to make.

Kernel: Definition

Given a TU game (N, v) and a partition Q of N, the Kernel is a set of allocations $x \in \mathbb{R}^n$ such that

1. $x \in I(Q)$
2. For every coalition $C \in Q$ and every pair of players $i, j \in E$,

$$\max_{\substack{C \subseteq N-j \\ i \in C}} e(C, x) = \max_{\substack{D \subseteq N-i \\ j \in D}} e(D, x)$$

Kernel: An Example

Consider the following four person game:

$$N = \{1, 2, 3, 4\}$$

$$Q = \{\{1, 2\}, \{3, 4\}\}$$

$$i = 1; \quad j = 2$$

$$I(Q) = \{(x_1, x_2, x_3, x_4) : x_1 \geq 0; x_2 \geq 0; x_3 \geq 0; x_4 \geq 0; x_1 + x_2 = v(12); \; x_3 + x_4 = v(34)\}$$

The Kernel of (N, v), with respect to Q will be the set of all allocations (x_1, x_2, x_3, x_4) from $I(Q)$ satisfying

$$\max_{\substack{C \subseteq \{1,3,4\} \\ i \in C}} e(C, x) = \max_{\substack{D \subseteq \{2,3,4\} \\ 2 \in D}} e(D, x)$$

The above leads to

$$\max\{e(1, x), e(13, x), e(14, x), e(134, x)\} = \max\{e(2, x), e(23, x), e(24, x), e(234, x)\}$$

Example : Apex Game

For the five player apex game considered earlier, the Kernel with the partition $\{\{1, 2, 3, 4, 5\}\}$ is $\{(\frac{3}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7},)\}$ where as with the partition $\{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$ the Kernel is $\{(\frac{1}{2}, \frac{1}{2}, 0, 0, 0)\}$.

4 Nucleolus

The Shapley value of a TU game is a solution concept that is primarily based on fairness in allocation. The nucleolus which is also unique allocation, like the Shapley value, is primarily based on bargaining considerations. This solution concept was proposed by David Schmeidler (1970) [3].

Recall that the excess of a coalition C wrt x is a measure of unhappiness of C with allocation x. Let us say we find an imputation that minimizes the largest among all excesses $e(C, x)$. This means the chosen imputation makes the most unhappy coalition as little unhappy as possible. The nucleolus is based on this idea and minimizes not only the level of unhappiness of the most unhappy coalition but also the levels of unhappiness of the second most unhappy coalition, third most unhappy coalition, etc.

Nucleolus: An Example and Definition

Consider a three player game [1] with $N = \{1, 2, 3\}$ and characteristic function defined by

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = 60;\ v(13) = 80;\ v(23) = 100$$

$$v(123) = 105$$

Given an allocation $x = (20, 35, 50)$, the excess values for the coalitions are as follows.

$$e(1, x) = -20;\ e(2, x) = -35;\ e(3, x) = -50$$

$$e(12, x) = 5;\ e(13, x) = 10;\ e(23, x) = 15$$

$$e(123, x) = 0$$

In the above, the largest excess is $e(23, x) = 15$. Let us try to reduce this by trying the allocation $y = (15, 35, 55)$ (say). Now we notice that

$$e(12, y) = 10;\ e(13, y) = 10;\ e(23, y) = 10$$

If we try to lower this any further, it would raise at least one other excess beyond 10. Thus the allocation y achieves the lowest excess for any coalition. In general, there may be multiple imputations which minimize the second largest excess to obtain a subset of the above set of imputations. Next, we minimize the third largest excess, fourth largest excess, etc. until, as shown in [3], we end up with a unique imputation. This unique imputation is called the *nucleolus*.

Consider any allocation $x = (x_1, \ldots, x_n)$ and let $e_k(x)$ be the largest excess generated by any coalition with allocation x. This means

$$|\{C \subseteq N : e(c, x) \geq e_k(x)\}| \geq k$$

$$|\{C \subseteq N : e(c, x) > e_k(x)\}| < k$$

Now, x belongs to *rmcore* (N, v) implies $e_1(x) \leq 0$. Suppose we denote by $J(k)$ the set of all imputations that minimize e_k where e_k is the k^{th} largest excess and by $I(N, v)$ the set of all imputations of (N, v). Then

$$J(1) = \overset{argmin}{x \in I(n, v)} \quad e_1(x)$$

If the core is non-empty, then $J(1) \subseteq$ core (N, v). We can define $J(2), J(3), \ldots, J(2^{|N|} - 1)$ as follows.

$$J(k) = \overset{argmin}{x \in J(k-1)} \quad e_k(x) \quad \text{for } k = 2, 3, \ldots, 2^{|N|} - 1$$

It was shown by Schmeidler [3] that the set $J(2^{|N|} - 1)$ is a singleton and this point is called the nucleolus of the TU game (N, v).

Note that when the core is non-empty, for any imputations in the core, all the excesses are zero or negative. For finding the nucleolus, we choose the imputation in the core which makes the least negative excess as negative as possible. Geometrically, the nucleolus is a point in the core which is as far inside the core as possible. We note a few other aspects about this solution concept. For details, the reader is referred to [1].

- The nucleolus always exists and is unique.
- If the core is non-empty, the nucleolus belongs to the core.
- The nucleolus always belongs to the bargaining set for the grand coalition. Also, it always belongs to the kernel.
- The nucleolus is not necessarily equal to the Shapley value. In fact, the nucleolus may allocate the surplus to players in a ruthless manner while Shapley value is built on the principle of fairness.
- The nucleolus can be computed by solving a series of linear programs.

5 The Gately Point

This solution concept was proposed by Dermot Gately (1974). Like the nucleolus, this is also based on the bargaining ability of the players. Suppose we have a TU game (N, v) with $N = \{1, 2, \ldots, n\}$. Let us focus on player i. If player i breaks away from the grand coalition, it might result in a loss (or gain) for the players. Suppose $x = (x_1, \ldots, x_n)$ is the original allocation for the grand coalition. Then the loss to the player i due to breakup is $x_i - v(\{i\})$. The joint loss to the rest of the players is

$$\sum_{j \neq i} x_j - v(N \backslash \{i\})$$

The disruption caused by player i breaking away can be measured in terms of so called *propensity to disrupt* which is defined as

$$d_i(x) = \frac{\sum_{j \neq i} x_j - v(N \backslash \{i\})}{x_i - v(\{i\})}$$

The Gately point is defined as an imputation that minimizes the maximum propensity to disrupt. It can be shown that minimizing the largest propensity to disrupt can be achieved by making the propensities to disrupt of all the players.

Gately Point: An Example

Consider a three player game with $N = \{1, 2, 3\}$ and

$$v(1) = v(2) = v(3) = 0$$
$$v(12) = 4;\ v(13) = 0;\ v(23) = 3$$
$$v(123) = 6$$

With respect to the allocation $x = (2, 3, 1)$, the propensities to disrupt are given by

$$d_1(x) = \frac{1}{2};\ d_2(x) = 1;\ d_3(x) = 1$$

If we try to equalize the propensities to disrupt, we end up with the allocation $y = (\frac{18}{11}, \frac{36}{11}, \frac{12}{11},)$ which yields

$$d_1(y) = d_2(y) = d_3(y) = \frac{5}{6}$$

The above allocation y turns out to be the Gately point. We make the following observations about this solution concept.

- Gately point does not necessarily belong to the core if the core is non-empty.
- The concept of Gately point can be extended by considering propensities to disrupt of coalitions rather than individual players. The resulting solution concept is called *disruptive nucleolus* which has been shown to belong to the core if the core is non-empty.

6 To Probe Further

This chapter relies heavily on the textbooks of Myerson [4] and Straffin [1]. These references may be looked into for more details. The above references also describe many other solution concepts such as Shapley-Shubik power index [5] and Banzhaf index [6]. The original references for the solution concepts discussed here are: stable sets [7], bargaining sets [8], Kernel [2], nucleolus [3], and Gately point [9].

7 Problems

1. Prove that the kernel of for the apex game having five players with the partition $\{\{1,2,3,4,5\}\}$ is $(\frac{3}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$ while with respect to the partition $\{\{1,2\},\{3\},\{4\},\{5\}\}$, the kernel is $(\frac{1}{2},\frac{1}{2},0,0,0)$. What will be the kernel with respect to the partition $\{\{1\},\{2,3,4,5\}\}$?
2. Show that the nucleolus of the apex game with five players is $(\frac{3}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7},\frac{1}{7})$.
3. Consider a three person superadditive game with $v(1) = v(2) = v(3) = 0; v(12) = a; v(13) = b; v(23) = c; v(123) = d$ where $0 \leq a, b, c \leq d$. Compute the nucleolus and Gately point for this game.
4. For the following three player game, compute the core, Shapley value, kernel, nucleolus, and Gately point.

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = 4; \quad v(13) = 0; \quad v(23) = 3; \quad v(123) = 6$$

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