Game Theory

Lecture Notes By

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July 2012

Further Topics in Mechanism Design

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`*. A more thorough version would be available soon in this space.*

1 Further Topics in Mechanism Design

Mechanism design is a rich area with a vast body of knowledge. So far in this chapter, we have seen essential aspects of game theory, followed by key results in mechanism design. We now provide a brief description of a few topics in mechanism design. The topics have been chosen, with an eye on their possible application to network economics problems of the kind discussed in the monograph. We have not followed any particular logical order while discussing the topics. We also caution the reader that the treatment is only expository. Pointers to the relevant literature are provided wherever appropriate.

1.1 Characterization of DSIC Mechanisms

We have seen that a direct revelation mechanism is specified as $\mathscr{D} = ((\Theta_i)_{i \in N}, f(.))$, where f is the underlying social choice function and Θ_i is the type set of agent i. A valuation function of each agent i, $v_i(.)$, associates a value of the allocation chosen by f to agent i, that is, $v_i : K \to \mathbb{R}$, where K is the set of project choices.

In the case of an auction for selling a single indivisible item, suppose each agent i has a valuation for the object $\theta_i \in [\underline{\theta_i}, \overline{\theta_i}]$. If agent i gets the object, $v_i(., \theta_i) = \theta_i$. Otherwise the valuation is zero. Thus for the agent i, the set of valuation functions over the set of allocations K can be written as $\Theta_i = [\underline{\theta_i}, \overline{\theta_i}]$. Thus Θ_i is single dimensional in this environment.

In a general setting, Θ_i may not be single dimensional. If we consider all real valued functions on X and allow each agent to have a valuation function to be any of these functions, we say Θ_i is unconstrained. Suppose $|K| = m$, then $\theta_i \in \Theta_i$ is an m-dimensional vector:

$$\theta_i = (\theta_{i_1}, \ldots, \theta_{i_j}, \ldots, \theta_{i_m}).$$

Note that θ_{i_j} will be the valuation of agent i if the j^{th} allocation from K is selected. In other words, $v_i(j) = \theta_{i_j}$. With such unconstrained type sets/valuation functions, an elegant characterization of

DSIC social choice functions has been obtained by Roberts [1]. The work of Roberts generalizes the the Green–Laffont Theorem (Theorem **??**) in the following way. Recall that the Green–Laffont Theorem asserts that an allocatively efficient and DSIC social choice function in the above unconstrained setting has to be necessarily a VCG mechanism. The result of Roberts asserts that all DSIC mechanisms are variations of the VCG mechanism. These variants are often referred to as the *weighted VCG mechanisms*. In a weighted VCG mechanism, weights are given to the agents and to the outcomes. The resulting social choice function is said to be an *affine maximizer*. The notion of an affine maximizer is defined below. Next we state the Roberts' Theorem.

Definition 1.1 *A social choice function f is called an affine maximizer if for some subrange $A' \subset X$, for some agent weights $w_1, w_2, \ldots, w_n \in \mathbb{R}^+$, and for some outcome weights $c_x \in \mathbb{R}$, and for every $x \in A'$, we have that*

$$f(\theta_1, \theta_2, \ldots, \theta_n) \in \arg\max_{x \in A'} (c_x + \sum_i w_i v_i(x)).$$

Theorem 1.1 (Roberts' Theorem) *If $|X| \geq 3$ and for each agent $i \in N$, Θ_i is unconstrained, then any DSIC social choice function f has nonnegative weights $w_1, w_2, \ldots, w_n$ (not all of them zero) and constants $\{c_x\}_{x \in X}$, such that for all $\theta \in \Theta$,*

$$f(\theta) \in \arg\max_{x \in X} \left\{ \sum_{i=1}^{n} w_i v_i(x) + c_x \right\}.$$

For a proof of this important theorem, we refer the reader to the article by Roberts [1]. Lavi, Mu'alem, and Nisan have provided two more proofs for the theorem — interested readers might refer to their paper [2] as well.

1.2 Dominant Strategy Implementation of BIC Rules

Clearly, dominant strategy incentive compatibility is stronger and much more desirable than Bayesian incentive compatibility. A striking reason for this is any Bayesian implementation assumes that the private information structure is common knowledge. It also assumes that the social planner knows a common prior distribution. In many cases, this requirement might be quite demanding. Also, a slight mis-specification of the common prior may lead the equilibrium to shift quite dramatically. This may result in unpredictable effects; for example it might cause an auction to to be highly nonoptimal.

A dominant strategy implementation overcomes these problems in a simple way since the equilibrium strategy does not depend upon the common prior distribution. We would therefore always wish to have a DSIC implementation. Since the class of BIC social choice functions is much richer than DSIC social choice functions, one would like to ask the question: Can we implement a BIC SCF as a DSIC rule with the same expected interim utilities to all the players? Mookherjee and Stefan [3] have answered this question by characterizing BIC rules that can be equivalently implemented in dominant strategies. When these sufficient conditions are satisfied, a BIC social choice function could be implemented without having to worry about a common prior. The article by Mookherjee and Stefan [3] may be consulted for further details.

1.3 Implementation in Ex-Post Nash Equilibrium

Dominant strategy implementation and Bayesian implementation are widely used for implementing a social choice function. There exists another notion of implementation, called ex-post Nash implementation, which is stronger than Bayesian implementation but weaker than dominant strategy implementation. This was formalized by Maskin [4]. Dasgupta, Hammond, and Maskin [5] generalized this to the Bayesian Nash implementation.

Definition 1.2 *A profile of strategies* $\left(s_1^*(.), s_2^*(.), \ldots, s_n^*(.)\right)$ *is an ex-post Nash equilibrium if for every* $\theta = (\theta_1, \ldots \theta_n) \in \Theta$*, the profile* $\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right)$ *is a Nash equilibrium of the complete information game defined by* $(\theta_1, \ldots, \theta_n)$*. That is, for all* $i \in N$ *and for all* $\theta \in \Theta$*, we have*

$$u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i) \geq u_i(s_i'(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i) \; \forall \; s_i' \in S_i.$$

In a Bayesian Nash equilibrium, the equilibrium strategy is played by the agents after observing their own private types and computing an expectation over others' types; it is an equilibrium only in the expected sense. On the other hand, in ex-post Nash equilibrium, even after the players are informed of the types of the other players, it is still a Nash equilibrium for each agent i to play an action according to $s_i^*(\cdot)$. This is called the *lack of regret* feature. That is, even if agents come to know about the others' types, the agent need not regret playing this action. Bayesian Nash equilibrium may not have this feature since the agents may want to revise their strategies after knowing the types of the other agents.

For example, consider the first price sealed bid auction with two bidders. Let $\Theta_1 = \Theta_2 = [0, 1]$ and θ_1 denote the valuation of the first agent and θ_2 that of the agent 2. It can be shown that it is a Bayesian Nash equilibrium for each bidder to bid according to the strategy $(b_1^*(\theta_1), b_2^*(\theta_2)) = (\frac{\theta_1}{2}, \frac{\theta_2}{2})$. Now suppose agent 1 is informed that the other agent values the object at 0.6. If agent 1 has a valuation of 0.8, say, it is not a Nash equilibrium for him to bid 0.4 even if agent 2 is still following a Bayesian Nash strategy.

Definition 1.3 *We say that the mechanism* $\mathscr{M} = ((S_i)_{i \in N}, g(\cdot))$ *implements the social choice function* $f(\cdot)$ *in ex-post Nash equilibrium if there is a pure strategy ex-post Nash equilibrium* $s^*(\cdot) = (s_1^*(\cdot), \ldots, s_n^*(\cdot))$ *of the game* Γ^b *induced by* $\mathscr{M}$ *such that*

$$g\left(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)\right) = f\left(\theta_1, \ldots, \theta_n\right) \; \forall \left(\theta_1, \ldots, \theta_n\right) \in \Theta.$$

Though ex-post implementation is stronger than Bayesian Nash implementation, it is still much weaker than dominant strategy implementation.

1.4 Interdependent Values

We have so far assumed that the private values or signals observed by the agents are independent of one another. This is a reasonable assumption in many situations. However, in the real world, there are environments where the valuation of agents might depend upon the information available or observed by the other agents. We will look at two examples.

Example 1 Consider an auction for an antique painting. There is no guarantee that the painting is an original one or a plagiarized version. If all the agents knew that the painting is not an original one, they would have a very low value for it independent of one another, whereas on the other hand, they would have a high value for it when it is a genuine piece of work. But suppose they have no knowledge about its authenticity. In such a case, if a certain bidder happens to get information about its genuineness, the valuations of all the other agents will naturally depend upon this signal (indicating the authenticity of the painting) observed by this agent.

Example 2 Consider an auction for oil drilling rights. At the time of the auction, buyers usually conduct geological tests and their private valuations depend upon the results of these tests. If a prospective bidder knew the results of the tests of the others, his own willingness to pay for the drilling rights would be modulated suitably based on the information available.

The interdependent private value models have been studied in the mechanism design literature. For example, there exists a popular model called the *common value model* (which we have already seen in Section **??**). As another example, consider a situation when a seller is trying to sell an indivisible good or a fixed quantity of a divisible good. The value of the received good for the bidders depends upon each others' private signals. Also, the private signals observed by the agents are interdependent of specified properties. In such a scenario, Cremer and McLean [6] have designed an auction that extracts a revenue from the bidders, which is equal to what could have been extracted when the actual signals of the bidders are known. In this auction, it is an ex-post Nash equilibrium for the agents to report their true types. This auction is interim individually rational but may not be ex-post individually rational.

1.5 Implementation Theory

Dominant strategy incentive compatibility ensures that reporting true types is a weakly dominant strategy equilibrium. Bayesian incentive compatibility ensures that reporting true types is a Bayesian Nash equilibrium. Typically, the Bayesian game underlying a given mechanism may have multiple equilibria, in fact, could have infinitely many equilibria. These equilibria typically will produce different outcomes. Thus it is possible that nonoptimal outcomes are produced by truth revelation.

The *implementation problem* addresses the above difficulty caused by multiple equilibria. The implementation problem seeks to design mechanisms in which all the equilibrium outcomes are optimal. This property is called the *weak implementation property*. If it also happens that every optimum outcome is also an equilibrium, we call the property as *full implementation property*. Maskin [4] provided a general characterization of Nash implementable social choice functions using a monotonicity property, which is now called *Maskin Monotonicity*. This property has a striking similarity to the property of independence of irrelevant alternatives, which we encountered during our discussion on Arrow's impossibility theorem (Section **??**). Maskin's work shows that Maskin monotonicity, in conjunction with another property called *no-veto-power* will guarantee that all Nash equilibria will produce an optimal outcome. His work has led to development of implementation theory. Dasgupta, Hammond, and Maskin [5] have summarized many important results in implementation theory, and they discuss incentive compatibility in detail. Maskin's results have now been generalized in many directions; for example, see the references in [7].

1.6 Computational Issues in Mechanism Design

We have seen several possibility and impossibility results in the context of mechanism design. While every possibility result is good news, there could be still be challenges involved in actually implementing a mechanism that is possible. For example, we have seen that the GVA mechanism (Example **??**) is an allocatively efficient and dominant strategy incentive compatible mechanism for combinatorial auctions. A major difficulty with GVA is the computational complexity involved in determining the allocation and the payments. Both the allocation and payment determination problems are NP-hard, being instances of the weighted set packing problem (in the case of forward GVA) or the weighted set covering problem (in the case of reverse GVA). In fact, if there are n agents, then in the worst case, the payment determination will involve solving as many as n NP-hard problems, so overall, as many as $(n+1)$ NP-hard problems will need to be solved for implementing the GVA mechanism. Moreover, approximately solving any one of these problems may compromise properties such as efficiency and/or incentive compatibility of the mechanism.

In mechanism design, computations are involved at two levels: first, at the agent level and secondly at the mechanism level [8, 9]. Complexity at the agent level involves strategic complexity (complexity of computing an optimal strategy) and valuation complexity (computation required to provide preference information within a mechanism). Complexity at the mechanism level includes communication complexity (how much communication is required between agents and the mechanism to compute an outcome) and winner determination complexity (computation required to determine an outcome given the strategies of the agents). Typically, insufficient computation leading to approximate solutions hinders mechanism design since properties such as incentive compatibility, allocative efficiency, individual rationality, etc., may be compromised. Novel algorithms and high computing power surely lead to better mechanisms.

For a detailed description of computational complexity issues in mechanism design, the reader is referred to the excellent survey articles [10, 8, 9].

2 To Probe Further

For a microeconomics oriented treatment of mechanism design, the readers are requested to refer to textbooks, such as the ones by Mas-Colell, Whinston, and Green [11] (Chapter 23); Green and Laffont [12]; and Laffont [13]. There is an excellent recent survey article by Nisan [14], which targets a computer science audience. There are many other informative survey papers on mechanism design — for example by Myerson [15], Serrano [16], and Jackson [17, 18]. The Nobel Prize website has a scholarly technical summary of mechanism design theory [7]. The recent edited volume on *Algorithmic Game Theory* by Nisan, Roughgarden, Tardos, and Vazirani [19] also has valuable articles related to mechanism design.

This chapter is not to be treated as a survey on auctions in general. There are widely popular books (for example, by Milgrom [20], Krishna [21], and Klemperer [22]) and surveys on auctions (for example, [23, 24, 25, 26, 8]) that deal with auctions in a comprehensive way.

A related area where an extensive amount of work has been carried out in the past decade is combinatorial auctions. Exclusive surveys on combinatorial auctions include the articles by de Vries and Vohra [10], Pekec and Rothkopf [27], and Narahari and Pankaj Dayama [28]. Cramton, Ausubel, and Steinberg [29] have brought out a comprehensive edited volume containing expository and survey articles on varied aspects of combinatorial auctions.

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