Game Theory

Lecture Notes By

Y. Narahari

Department of Computer Science and Automation

Indian Institute of Science

Bangalore, India

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Properties of Social Choice Functions

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

1 Properties of Social Choice Functions

We have seen that a mechanism provides a solution to both the preference elicitation problem and preference aggregation problem, if the mechanism can implement the desired social choice function $f(\cdot)$. It is obvious that some SCFs are implementable and some are not. Before we look into the question of characterizing the space of implementable social choice functions, it is important to know which social choice function ideally a social planner would wish to implement. In this section, we highlight a few properties of an SCF that ideally a social planner would wish the SCF to have.

1.1 Ex-Post Efficiency

Definition 1.1 (Ex-Post Efficiency) *The SCF $f : \Theta \to X$ is said to be ex-post efficient (or Paretian) if for every profile of agents' types, $\theta \in \Theta$, the outcome $f(\theta)$ is a Pareto optimal outcome. The outcome $f(\theta_1, \ldots, \theta_n)$ is Pareto optimal if there does not exist any $x \in X$ such that:*

$$u_i(x, \theta_i) \geq \ u_i(f(\theta), \theta_i) \ \forall \ \ i \in N \ \text{ and } \ u_i(x, \theta_i) > \ u_i(f(\theta), \theta_i) \ \text{ for some } \ i \in N.$$

Example 1 (Supplier Selection Problem) Consider the supplier selection problem (Example **??**). Let the social choice function f be given by

$$\begin{aligned} f(a_1, a_2) &= x \\ f(a_1, b_2) &= x. \end{aligned}$$

The outcome $f(a_1, a_2) = x$ is Pareto optimal since the other outcomes y and z are such that

$$\begin{aligned} u_1(y, a_1) &< u_1(x, a_1) \\ u_1(z, a_1) &< u_1(x, a_1). \end{aligned}$$

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Thus SCF 1 is ex-post efficient.

Example 2 (Procurement of a Single Indivisible Item) We have looked at three social choice functions, SCF-PROC1, SCF-PROC2, SCF-PROC3, in the previous section. One can show that all these SCFs are ex-post efficient.

1.2 Dictatorship in SCFs

We define this through a dictatorial social choice function.

Definition 1.2 (Dictatorship) *A social choice function $f : \Theta \to X$ is said to be dictatorial if there exists an agent d (called dictator) who satisfies the following property:*

$$\forall\, \theta \in \Theta,\ \ f(\theta) \text{ is such that } u_d(f(\theta), \theta_d) \geq u_d(x, \theta_d)\ \ \forall\, x \in X.$$

A social choice function that is not dictatorial is said to be nondictatorial .

In a dictatorial SCF, every outcome that is picked by the SCF is such that it is a most favored outcome for the dictator.

Example 3 (Supplier Selection Problem) Let the social choice function f be given by

$$f(a_1, a_2) = x;\ \ f(a_1, b_2) = x.$$

It is easy to see that agent 1 is a dictator and hence this is a dictatorial SCF. On the other hand, consider the following SCF:

$$f(a_1, a_2) = x;\ \ f(a_1, b_2) = y.$$

One can verify that this is not a dictatorial SCF.

1.3 Individual Rationality

Individual rationality is also often referred to as voluntary participation property. Individual rationality of a social choice function essentially means that each agent gains a nonnegative utility by participating in a mechanism that implements the social choice function. There are three stages at which individual rationality constraints (also called participation constraints) may be relevant in a mechanism design situation.

1.3.1 Ex-Post Individual Rationality

These constraints become relevant when any agent i is given a choice to withdraw from the mechanism at the ex-post stage, that is, after all the agents have announced their types and an outcome in X has been chosen. Let $\overline{u_i}(\theta_i)$ be the utility that agent i receives by withdrawing from the mechanism when his type is θ_i. Then, to ensure agent i's participation, we must satisfy the following *ex-post participation (or individual rationality) constraints*

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq \overline{u_i}(\theta_i)\ \ \forall\, (\theta_i, \theta_{-i}) \in \Theta.$$

1.3.2 Interim Individual Rationality

Let the agent i be allowed to withdraw from the mechanism only at an interim stage that arises after the agents have learned their type but before they have chosen their actions in the mechanism. In such a situation, the agent i will participate in the mechanism only if his interim expected utility $U_i(\theta_i|f) = E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i]$ from social choice function $f(\cdot)$, when his type is θ_i, is greater than $\overline{u_i}(\theta_i)$. Thus, *interim participation (or individual rationality) constraints* for agent i require that

$$U_i(\theta_i|f) = E_{\theta_{-i}}[u_i(f(\theta_i, \theta_{-i}), \theta_i)|\theta_i] \geq \overline{u_i}(\theta_i) \ \ \forall \, \theta_i \in \Theta_i.$$

1.3.3 Ex-Ante Individual Rationality

Let agent i be allowed to refuse to participate in a mechanism only at ex-ante stage, that is, before the agents learn their type. In such a situation, the agent i will participate in the mechanism only if his ex-ante expected utility $U_i(f) = E_\theta[u_i(f(\theta_i, \theta_{-i}), \theta_i)]$ from social choice function $f(\cdot)$ is at least $E_{\theta_i}[\overline{u_i}(\theta_i)]$. Thus, *ex-ante participation (or individual rationality) constraints* for agent i require that

$$U_i(f) = E_\theta[u_i(f(\theta_i, \theta_{-i}), \theta_i)] \geq E_{\theta_i}[\overline{u_i}(\theta_i)].$$

The following proposition establishes a relationship among the three different participation constraints discussed above. The proof is left as an exercise.

Proposition 1.1 *For any social choice function $f(\cdot)$, we have*

$$f(\cdot) \textit{ is ex-post IR} \Rightarrow f(\cdot) \textit{ is interim IR} \Rightarrow f(\cdot) \textit{ is ex-ante IR.}$$

1.4 Efficiency

We have seen the notion of ex-post efficiency already. Depending on the epoch at which we look into the game, we have three notions of efficiency, on the lines of individual rationality. These notions were introduced by Holmstorm and Myerson [**?**]. Let F be any collection of social choice functions that are of interest.

Definition 1.3 (Ex-Ante Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-ante efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties:*

$$\begin{aligned} E_\theta[u_i(\hat{f}(\theta), \theta_i)] &\geq E_\theta[u_i(f(\theta), \theta_i)] \ \ \forall \, i = 1, \ldots, n, \\ E_\theta[u_i(\hat{f}(\theta), \theta_i)] &> E_\theta[u_i(f(\theta), \theta_i)] \text{ for some } i. \end{aligned}$$

Definition 1.4 (Interim Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is interim efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties:*

$$\begin{aligned} E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] &\geq E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \ \ \forall \, i = 1, \ldots, n, \ \forall \, \theta_i \in \Theta_i, \\ E_{\theta_{-i}}[u_i(\hat{f}(\theta), \theta_i)|\theta_i] &> E_{\theta_{-i}}[u_i(f(\theta), \theta_i)|\theta_i] \text{ for some } i \text{ and some } \theta_i \in \Theta_i. \end{aligned}$$

Definition 1.5 (Ex-Post Efficiency) *For any given set of social choice functions F, and any member $f(\cdot) \in F$, we say that $f(\cdot)$ is ex-post efficient in F if there is no other $\hat{f}(\cdot) \in F$ having the following two properties:*

$$\begin{aligned} u_i(\hat{f}(\theta), \theta_i) &\geq u_i(f(\theta), \theta_i) \;\; \forall\, i = 1, \ldots, n, \; \forall\, \theta \in \Theta, \\ u_i(\hat{f}(\theta), \theta_i) &> u_i(f(\theta), \theta_i) \text{ for some } i \text{ and some } \theta \in \Theta. \end{aligned}$$

Using the above definition of ex-post efficiency, we can say that a social choice function $f(\cdot)$ is ex-post efficient in the sense of Definition 1.1 if and only if it is ex-post efficient in the sense of Definition 1.5 when we take $F = \{f : f \text{ is a mapping from } \Theta \text{ to } X\}$.

The following proposition establishes a relationship among these three different notions of efficiency.

Proposition 1.2 *Given any set of feasible social choice functions F and $f(\cdot) \in F$, we have*

$$f(\cdot) \textit{ is ex-ante efficient} \Rightarrow f(\cdot) \textit{ is interim efficient} \Rightarrow f(\cdot) \textit{ is ex-post efficient.}$$

For a proof of the above proposition, refer to Proposition 23.F.1 of [**?**]. Also, compare the above proposition with the Proposition 1.1.

2 Problems

1. Let $N = \{1, 2\}$; $\Theta_1 = \{a_1, b_1\}$; $\Theta_2 = \{a_2, b_2\}$; $X = \{x, y, z\}$; and

$$u_1(x, a_1) = 100; \;\; u_1(y, a_1) = 50; \;\; u_1(z, a_1) = 0$$

$$u_1(x, b_1) = 50; \;\; u_1(y, b_1) = 100; \;\; u_1(z, b_1) = 40$$

$$u_2(x, a_2) = 0; \;\; u_2(y, a_2) = 50; \;\; u_2(z, a_2) = 100$$

$$u_2(x, b_2) = 50; \;\; u_2(y, b_2) = 30; \;\; u_2(z, b_2) = 100$$

   For the above environment, suggest a social choice function in each case listed below (EPE - expost efficient; BIC - Bayesian Incentive Compatible; DSIC - dominant strategy incentive compatible; D- Dictatorial; ND - Non-dictatorial).

   - An SCF which is EPE, DSIC, D
   - An SCF which is EPE, DSIC, ND
   - An SCF which is not EPE but is DSIC, ND
   - An SCF which is EPE, BIC, but not DSIC
   - An SCF which is EPE, not BIC, not DSIC

2. A mother wants to distribute a cake equally between two of her children. She designs the following mechanism: one of the children will cut the cake and the other child will get the first chance to pick up the cake. The mechanism ensures that the social choice function (namely the cake is distributed equally between the two children) is implemented in dominant strategies. Now consider the case where there are 3 children instead of 2. For this problem, suggest (i) a mechanism which will implement the SCF in dominant strategies and (ii) a mechanism which will implement the SCF in Nash equilibrium but not in dominant strategies. Prove your results with simple, brief, logical arguments.

3. Consider a procurement auction for a single indivisible good with a buyer (agent 0) and n suppliers $(1, 2, \ldots, n)$. Each supplier submits a sealed bid and whoever submits the smallest bid will win the auction. If multiple players submit the same bid, then the agent with the lowest index among them is announced as the winner. The winning agent will receive a sum equal to the bid. Assume that the bids from the agents are independent draws from the uniform distribution on [0,1]. For this scenario, under a quasi-linear setting, write down the social choice function being implemented. Is the social choice function ex-post efficient? What is the underlying mechanism. Write down the components of the Bayesian game induced by this mechanism.

4. Let $f : \Theta_1 \times \ldots \times \Theta_n \to X$ be a social choice function such that $\forall \theta \in \Theta$,

$$f(\theta) \in \begin{matrix} argmax \\ x \in X \end{matrix} \left\{ \sum_{i=1}^{n} u_i(x, \theta_i) \right\}$$

Then show that $f(.)$ is ex-post efficient.

5. Consider the social choice function

$$f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$$

such that

$$\begin{aligned} y_0(\theta) &= 0 \quad \forall \theta \in \Theta & (1) \\ y_1(\theta) &= 1 \quad \text{if} \theta_1 \geq \theta_2 & (2) \\ &= 0 \quad \text{otherwise} & (3) \\ y_2(\theta) &= 1 \quad \text{if} \theta_1 \leq \theta_2 & (4) \\ &= 0 \quad \text{otherwise} & (5) \\ t_1(\theta) &= -y_1(\theta)\theta_1 & (6) \\ t_2(\theta) &= -y_2(\theta)\theta_2 & (7) \\ t_0(\theta) &= -(t_1(\theta) + t_2(\theta)) & (8) \end{aligned}$$

Show that f is ex-post efficient.

6. Show that the social choice function representing the following situation is ex-post efficient. There are n agents and an indivisible good is to be allocated to one of them. The good is allocated to an agent having highest valuation and the total amount received by all the agents together is zero.

7. Show by means of an example that when the buyer and seller in a bilateral trade setting both have a discrete set of possible valuation, social choice functions may exist that are Bayesian incentive compatible, ex-post efficient, and individually rational. [*Hint:* It suffices to let each have two possible types.]

8. Consider a seller who is faced with a single buyer. The type set of the buyer is $\Theta = \{0, 1, 2\}$. The set of outcomes is $X = \{a, b, c\}$. The valuation function of the buyer is $v(a, \theta) = 0 \ \forall \theta \in \Theta$; $v(b, \theta) = \frac{\theta}{2} \ \forall \theta \in \Theta$; $v(c, \theta) = \theta \ \forall \theta \in \Theta$. Write down all incentive compatible social choice functions for this setting. Note that there is only one agent here.