Game Theory

Lecture Notes By

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Subgame Perfect Equilibrium

Note: *This is a only a draft version, so there could be flaws. If you find any errors, please do send email to* `hari@csa.iisc.ernet.in`. *A more thorough version would be available soon in this space.*

We have studied extensive form games briefly in Chapter 2. We now revisit this important class of games and introduce a key solution concept called subgame perfect equilibrium. We start with two examples. In this chapter, we stick to extensive form games with perfect information.

1 Two Examples of Extensive Form Games

1.1 Example 1: Entry Game

In this game, there are two players, 1 and 2. Player 1 is called *challenger* and player 2 is called *incumbent*. Figure 1 shows the game tree.

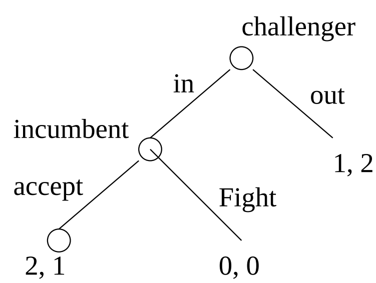

Figure 1: Game tree for entry game

Player 1 (challenger) either decides to challenge the incumbent (action: in) or drops out (action: out). Player 2 (incumbent) either decides to fight or accommodate the challenger in case the challenger decides to challenge the incumbent. The respective payoffs are shown in the game tree. For this game, we have

$$
\begin{aligned}
&N = \{1, 2\}; \quad A_1 = \{\text{in, out}\}; \quad A_2 = \{\text{accept, fight}\} \\
&\mathcal{H} = \{(\text{in, accept}), (\text{in, fight}), (\text{out})\} \\
&S_{\mathcal{H}} = \{\epsilon, (\text{in})\} \\
&P(\epsilon) = 1 \\
&P(\text{in}) = 2
\end{aligned}
$$

$$
\begin{aligned}
&u_1(\text{in, accept}) = 2; \quad u_1(\text{in, fight}) = 0; \quad u_1(\text{out}) = 1 \\
&u_2(\text{in, accept}) = 1; \quad u_2(\text{in, fight}) = 0; \quad u_2(\text{out}) = 2
\end{aligned}
$$

Note that the action sets A_1 and A_2 can be deduced from the terminal histories and the player function.

1.2 Example 2

Consider the game tree shown in Figure 2. For this game, it is easy to see that

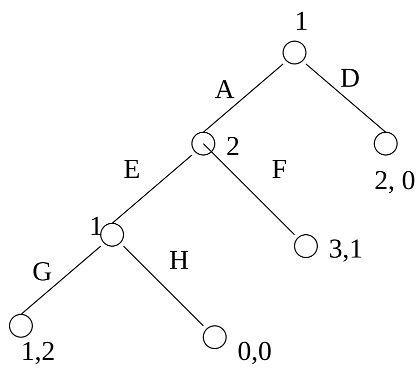

Figure 2: Extensive form game of Example 2

$$N = \{1, 2\}; \quad A_2 = \{C, D, G, H\}; \quad A_2 = \{E, F\}$$

The terminal histories are given by

$$\mathcal{H} = \{(C, E, G)\}, (C, E, H), (C, F), (D)\}$$

The proper subhistories of terminal histories are given by

$$S_{\mathcal{H}} = \{\epsilon, (C), (C, E)\}$$

The player function is given by

$$P(\epsilon) = 1; \quad P(C) = 2; \quad P(C, E) = 1$$

The utility functions are given by

$$
\begin{aligned}
&u_1(C, E, G) = 1; \quad u_1(C, E, H) = 0; \\
&u_1(C, F) = 3; \quad u_1(D) = 2; \\
&u_2(C, E, G) = 2; \quad u_2(C, E, H) = 0; \\
&u_2(C, F) = 1; \quad u_2(D) = 0.
\end{aligned}
$$

In chapter 2, we have seen the notion of a strategy which is a complete contingent plan of action of a player. If S_1 and S_2 are the sets of strategies of players 1 and 2 respectively, it is easy to see that

$$
\begin{aligned}
S_1 &= \{CG, CH, DG, DH\} \\
S_2 &= \{E, F\} \qquad (1)
\end{aligned}
$$

The set of strategy profiles, $S_1 \times S_2$, is given by

$$S_1 \times S_2 = \{(CG, E), (CG, F), (CH, E), (CH, F), (DG, E), (DG, F), (DH, E), (DH, F)\}$$

Note that a strategy profile uniquely determines a terminal history. For example, the profile (CG, E) corresponds to the terminal history (C,E,G); the profile (CG,F) corresponds to the terminal history (C,F), etc. This leads to the following definition.

Definition 1 (Outcome) *Given an extensive form game TT and a strategy profile $s = (s_1, \ldots, s_n)$ in the game, the terminal history corresponding to the strategy profile s is called the* outcome *of of s and is denoted by $O(s)$.*

2 The Notion of a Subgame

Given an extensive form game Γ and a non-terminal history h, the *subgame* following h is the part of the game that remains after h has occurred. Figure 3 shows the only proper subgame of the entry game of Figure 1. Figures 4 and 5 shows the two proper subgames of the game of Figure 2.

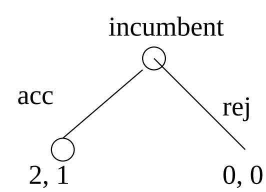

Figure 3: Subgame of entry game corresponding to history (in)

3 Nash Equilibrium and Subgame Perfect Equilibrium

The notion of Nash equilibrium follows immediately via strategic form game representation of extensive form games. We can formally define Nash equilibrium (pure strategy) as follows.

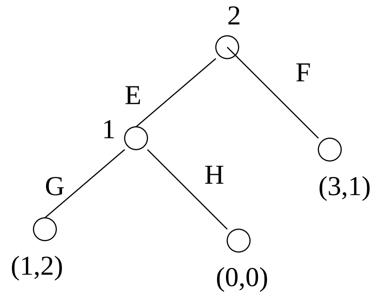

Figure 4: Subgame of Example 2 with history (C)

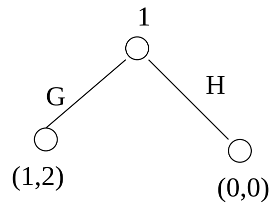

Figure 5: Subgame of Example 2 with history (C,E)

Definition 2 *Given an extensive form game* $\Gamma =< N, (A_i), \mathcal{H}, P, (u_i) >$*, a strategy profile* $s^* = (s_1^*, \ldots, s_n^*)$ *is called a pure strategy Nash equilibrium if* $\forall i \in N$*,*

$$u_i(O(s_i^*, s_{-i}^*)) \geq u_i(O(s_i, s_{-i}^*)) \;\; \forall s_i \in S_i$$

where S_i *is the set of all strategies of player* i $(i = 1, 2, \ldots, n)$*.*

As an immediate example, consider the entry game (Figure 1). The payoff matrix of strategic form game equivalent is given by

	2	
1	accept	fight
in	1,1	0,0
out	1,2	1,2

It is clear that both (in, accept) and (out, fight) are pure strategy Nash equilibria.

Note that the Nash equilibrium (in, accept) is intuitive while the Nash equilibrium (out, fight) looks illogical. However, the latter is because the notion of Nash equilibrium in extensive form games ignores the sequential structure of the extensive games. Nash equilibrium simply treats strategies as choices made once and for all before play begins.

Example 1 *In the extensive form game of Figure 2, recall the strategy sets of players 1 and 2:*

$$S_1 = \{CG,\ CG,\ DG,\ DH\}$$
$$S_2 = \{E,\ F\}$$

The payoff matrix of the equivalent strategic form game is given by

	2	
1	*E*	*F*
CG	*1,2*	*3,1*
CH	*0,0*	*3,1*
DG	*2,0*	*2,0*
DH	*2,0*	*2,0*

There are three pure strategy Nash equilibria here, namely (DH, E), (DG, E), and (CH, F).

3.1 Subgame Perfect Equilibrium

The notion of subgame perfect equilibrium (SGPE) takes into account every possible history in the game and ensures that each players strategy is optimal given the strategies of the other players, not only at the start of the game but after every possible history. The following is a formal definition.

Definition 3 (Subgame Perfect Equilibrium) . *Given an extensive form game* $\Gamma =< N, (A_i), \mathcal{H}, P, (u_i) >$, *a strategy profile* $s^* = (s_1^*, \ldots, s_n^*)$ *is a SGPE if* $\forall i \in N$,

$$u_i(O_h(s_i^*, s_{-i}^*)) \geq u_i(O_h(s_i, s_{-i}^*)) \;\; \forall h \in \{x \in s_{\mathcal{H}} : P(x) = i\}; \;\; \forall s_i \in S_i$$

Example 2 *In the entry game (Figure 1), we have seen that both (in, accept) and (out, fight) are both Nash equilibria. However only the profile (out, fight) is not an SGPE because the action "fight" is not optimal for player 2 in the subgame corresponding to history (in). The rationale is that in case the challenger deviates from (out, fight) and plays "in", then player 2 is at a disadvantage since "out" is not his best response.*

In the case of extensive game shown in Figure 2, we have seen that the profiles (DH, E), (DG, E), and (CH, F) are all Nash equilibria. However, only the profile (DG, E) is a SGPE.

3.2 NE versus SGPE

From the definition of SGPE, it is clear that SGPE is a strategy profile that induces a Nash equilibrium in every subgame of the game. Thus an SGPE is always a Nash equilibrium whereas the converse is definitely not true as we have already seen in the examples.

In a Nash equilibrium of an extensive game, each player's strategy is optimal given the strategies of the other players in the whole game. It may not be optimal in every subgame. However it will be optimal in any subgame that is reached when the players follow the Nash equilibrium strategies. On the other hand, an SGPE is such that each player's strategy is optimal in every possible history that may or may not occur if the players follow their strategies.

In a Nash equilibrium, each player has long experience of playing with other players and has correct beliefs about the actions of the other players. She believes they will not deviate from these actions and given these beliefs of wisdom, the Nash equilibrium strategy is optimal.

A subgame perfect equilibrium does not make such assumptions about the actions of the other players. Each player, even if on rare occasions, may deviate from SGPE actions. Each player forms correct beliefs about other players strategies and knows how the SGPE provides superior insurance against deviation by other players than a Nash equilibrium.

4 To Probe Further

Most of the material in this chapter owes to the excellent discussion in the book by Osborne [1]. The reader is referred to that book for more details. Myerson's book [2] also has a detailed discussion of extensive form games.

5 Problems

1. Find the Nash equilibria and SGPE of the following game.

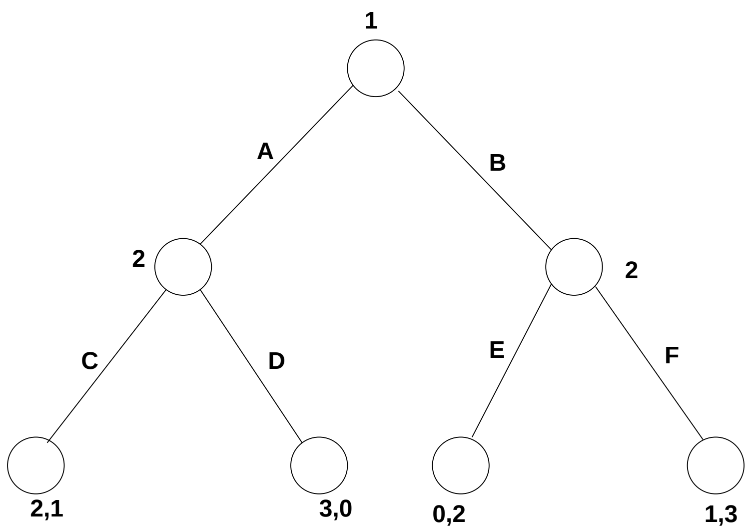

Figure 6: Game tree for Problem 1

2. Consider the following game called the ultimatum game. There are two players 1 and 2. Player 1 offers to player 2 an amount $0 \leq x \leq c$ where c is a constant. Player 2 either accepts this offer or rejects the offer. If player 2 accepts, then 2 receives x and 1 receives $c - x$. If player 2 rejects the offer, then both players end up with zero payoff. Write down the game tree, compute all Nash equilibria, and compute all SGPE.

3. For the matching pennies game with observation, compute all Nash equilibria and all SGPE.

4. For the sequential Prisoner's dilemma problem where player 1 moves first and then player 2, compute all Nash equilibria and all SGPE.

References

[1] Martin J. Osborne. *An Introduction to Game Theory.* The MIT Press, 2003.

[2] Roger B. Myerson. *Game Theory: Analysis of Conflict.* Harvard University Press, Cambridge, Massachusetts, USA, 1997.