Design of an Optimal Auction for Sponsored Search Auctions

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Abstract

In this paper, we first describe a framework to model the sponsored search auction on the web as a mechanism design problem. Using this framework, we design a novel auction which we call the OPT (optimal) auction. The OPT mechanism maximizes the search engine's expected revenue while achieving Bayesian incentive compatibility and individual rationality of the advertisers. We show that the OPT mechanism is superior to two of the most commonly used mechanisms for sponsored search namely (1) GSP (Generalized Second Price) and (2) VCG (Vickrey-Clarke-Groves). We then show an important revenue equivalence result that the expected revenue earned by the search engine is the same for all the three mechanisms provided the advertisers are symmetric and the number of sponsored slots is strictly less than the number of advertisers.

1 Introduction

The advertiser-supported web site is one of the successful business models in the emerging web landscape. In today's web advertising industry, *Search Ads* constitute the highest revenue generating model among all Internet advertising formats. In a typical search engine like Google, the results of a search query has two different stacks side by side - the left stack contains the links that are most relevant to the query term and the right stack contains the sponsored links. Typically, a number of merchants (advertisers) are interested in advertising alongside the search results of a keyword. However, the number of slots available to display the sponsored links is limited. Therefore, against every search performed by the user, the search engine faces the problem of matching the advertisers to the slots. In addition, the search engine also needs to decide on a price to be charged to each advertiser.

In this paper, we are interested in studying appropriate mechanisms for sponsored search auction and investigate their performance.

1.1 Related Literature

The motivation for our work comes from several recent research articles. The work of [2] investigates the Generalized Second Price (GSP) mechanism for sponsored search auction under *static settings*. Our approach generalizes their analysis to the more realistic case of incomplete information through a detailed analysis of the induced Bayesian game. Another strand of work which is closely related to ours is due to [5]. The objective of this paper is to clarify the incentive, efficiency, and revenue properties of the two popular slot auctions - first price and second price, under settings of incomplete and complete information. The work does not attempt to derive any optimal mechanism. In [3], author studies the allocation mechanisms under a setting in which the advertisers have a consistent ranking of advertising positions but different rates of decrease in absolute valuation. The model and underlying assumptions of this paper are quite different than ours. A recent paper [8] analyzes the equilibria of an assignment game that arises in the context of Ad auctions. In another related work by [1], the authors design a simple truthful auction for a general class of ranking functions that includes direct ranking and revenue ranking.

1.2 Contributions and Outline

In Section 2, we first develop a framework to model the sponsored search auction problem as a mechanism design problem. We then pursue the objective of designing a mechanism that satisfies the following requirements, which we believe are practical requirements for any sponsored search auction: (1) revenue maximization for the search engine; (2) individual rationality for the advertisers; and (3) Bayesian incentive compatibility of the mechanism. Motivated by this, in Section 3, we propose a new mechanism which we call the *Optimal (OPT)* mechanism. We then analyze the OPT mechanism assuming symmetric advertisers. Our analysis shows that the expected revenue earned by the search engine is the same for all the three mechanisms (GSP,

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0-7695-2913-5/07 $25.00 © 2007 IEEE

VCG, and OPT) provided the advertisers are symmetric and the number of sponsored slots is strictly less than the number of advertisers.

2 Sponsored Search Auction as a Mechanism Design Problem in Linear Environment

Consider a search engine that has received a query from an Internet user and it faces the problem of selling its advertising space among the available advertisers for this particular query word. We make the following assumptions.

(1) There are n advertisers interested in this particular keyword and $N = \{1, 2, \ldots, n\}$ represents the set of these advertisers. Also, there are m slots available with the search engine to display the Ads and $M = \{1, 2, \ldots, m\}$ represents the set of these advertising slots.

(2) α_{ij} is the probability that a user will click on the i^{th} advertiser's Ad if it is displayed in the j^{th} position (slot), where the first position refers to the top most position. We assume that the following condition is satisfied.

$$1 \geq \alpha_{i1} \geq \alpha_{i2} \geq \ldots \geq \alpha_{im} \geq 0 \; \forall i \in N \qquad (1)$$

Note here that we are assuming that click probability α_{ij} does not depend on which other advertiser has been allocated to what other position.

(3) Each advertiser precisely knows the value derived out of each click performed by the user on his Ad but does not know the value derived out of a single user-click by the other advertisers. Note that this value should be independent of the position of the Ad and should only depend on whether or not a user clicks on the Ad. Formally, this is modeled by supposing that advertiser i observes a parameter, or signal θ_i that represents his value for each user click. The parameter θ_i is referred to as advertiser i's *type*. The set of possible types of advertiser i is denoted by Θ_i.

(4) For advertiser i, $i = 1, 2, \ldots, n$, there is some probability distribution $\Phi_i(\cdot)$ from which he draws his valuation θ_i. Let $\phi_i(\cdot)$ be the corresponding PDF. We assume that the θ_i takes values from a closed interval $\left[\underline{\theta_i}, \overline{\theta_i}\right]$ of the real line. That is, $\Theta_i = \left[\underline{\theta_i}, \overline{\theta_i}\right]$. We also assume that any advertiser's valuation is statistically independent from any other advertiser's valuation. That is, $\Phi_i(\cdot), i = 1, 2, \ldots, n$ are mutually independent. This is the classical *independent private values assumption.*

(5) Each advertiser i is rational and intelligent and this fact is modeled by assuming that the advertisers always try to maximize a utility function $u_i : X \times \Theta_i \to \mathbb{R}$, where X is the set of outcomes which will be defined shortly.

(6) The probability distribution functions $\Phi_i(\cdot)$, the type sets $\Theta_1, \ldots, \Theta_n$, and the utility functions $u_i(\cdot)$ are assumed to be common knowledge among the advertisers. Note that utility function $u_i(\cdot)$ of advertiser i depends on both the outcome x and the type θ_i. The type θ_i is not a common knowledge; but by saying that $u_i(\cdot)$ is common knowledge we mean that for any given type θ_i, the auctioneer (that is, search engine in this case) and every other advertiser can evaluate the utility function of advertiser i.

A sponsored search auction can be viewed as an *indirect mechanism* $\mathscr{M} = ((B_i)_{i \in N}, g(\cdot))$, where $B_i \subset \mathbb{R}^+$ is the set of bids that an advertiser i can report to the search engine and $g(\cdot)$ is an allocation and payment rule. Note, if we assume that for each advertiser i, the set of bids B_i is the same as the type set Θ_i, then the indirect mechanism $\mathscr{M} = ((B_i)_{i \in N}, g(\cdot))$ becomes a direct revelation mechanism $\mathscr{D} = ((\Theta_i)_{i \in N}, f(\cdot))$, where $f(\cdot)$ becomes the allocation and payment rule. In the rest of this paper, we will assume that $B_i = \Theta_i \; \forall \; i = 1, \ldots, n$. Thus, we regard a sponsored search auction as a direct revelation mechanism. The various components of a typical sponsored search mechanism design problem are listed below.

1. An outcome in the case of sponsored search auction may be represented by a vector $x = (y_{ij}, p_i)_{i \in N, j \in M}$, where y_{ij} is the probability that advertiser i is allocated the slot j and p_i denotes the price-per-click charged from the advertiser i. The set of feasible alternatives is then

$$X = \Big\{ (y_{ij}, p_i)_{i \in N, j \in M} \Big| y_{ij} \in [0,1] \; \forall i \in N, \; \forall j \in M, \; \textstyle\sum_{i=1}^{n} y_{ij} \leq 1 \; \forall j \in M, \sum_{j=1}^{m} y_{ij} \leq 1 \; \forall i \in N, \; p_i \geq 0 \; \forall i \in N \Big\}$$

2. $u_i(x, \theta_i) = \left(\sum_{j=1}^{m} y_{ij} \alpha_{ij} \right) (\theta_i - p_i)$

3. $f(b) = (y_{ij}(b), p_i(b))_{i \in N, j \in M}$, where $b = (b_1, \ldots, b_n)$ is a bid vector of the advertisers. The functions $y_{ij}(\cdot)$ form the allocation rule and the functions $p_i(\cdot)$ form the payment rule.

Linear Environment

Through a slight modification in the definition of allocation rule, payment rule, and utility functions, we can show that sponsored search auction is indeed a mechanism in linear environment. To transform the underlying environment to a linear one, we redefine the allocation and payment rule as below.

$$f(b) = (y(b), t_i(b))_{i \in N, j \in M}$$

where $y(b) = (y_{ij}(b))_{i \in N, j \in M}$ and $t_i(b) = \left(\sum_{j=1}^{m} y_{ij}(b) \alpha_{ij} \right) p_i(b)$. The quantity $t_i(b)$ can be viewed as the average payment made by the advertiser i to the search engine against every search query received by the search engine and when the bid vector of the advertisers is $b = (b_1, \ldots, b_n)$.

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Now, we can rewrite the utility functions in following manner

$$u_i(f(b),\theta_i) \quad = \quad \theta_i v_i(y(b)) - t_i(b)$$

where $v_i(y(b)) = \left(\sum_{j=1}^{m} y_{ij}(b)\alpha_{ij}\right)$. The quantity $v_i(y(b))$ can be interpreted as the probability that advertiser i will receive a user click whenever there is a search query received by the search engine and when the bid vector of the advertisers is $b = (b_1,\ldots,b_n)$. Now, it is easy to verify that the underlying environment is *linear.*

3 The OPT Mechanism for Sponsored Search Auction

Both the commonly used mechanisms GSP and VCG can be shown to be individually rational [4]. However, we can show that the GSP mechanism is not Bayesian incentive compatible by following a line of attack similar to the one used by McAfee and McMillan [6] to compute the equilibrium bidding strategy of the buyers during the auction of a single indivisible good. The VCG mechanism, on the other hand, is dominant strategy incentive compatible, by design itself; however, it is known to lead to significantly high levels of incentives to be paid to the advertisers. This brings down the revenue of the search engine.

The above two factors motivate us to look for a mechanism for sponsored search auction that maximizes the revenue to the search engine subject to individual rationality and Bayesian incentive compatibility. Myerson [7] first studied such an auction mechanism in the context of selling a single indivisible good. Our goal in this paper is to compute the allocation and payment rule $f(\cdot)$ that results in an optimal mechanism for the sponsored search auction. This calls for extending Myerson's optimal auction to the case of the sponsored search auction. We follow a line of attack which is similar to that of [7].

3.1 Allocation Rule

It is convenient to define

- $\overline{v}_i(b_i) = E_{\theta_{-i}}[v_i(y(b_i,\theta_{-i}))]$ is the probability that advertiser i will receive a user click if he bids an amount b_i and all the advertisers $j \neq i$ bid their true types.

- $U_i(\theta_i) = \theta_i\overline{v}_i(\theta_i) - \overline{t}_i(\theta_i)$ gives advertiser i's expected utility from the mechanism conditional on his type being θ_i when he and all other advertisers bid their true types.

The problem of designing an optimal mechanism for the sponsored search auction can now be written as one of choosing functions $y_{ij}(\cdot)$ and $U_i(\cdot)$ to solve:

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_{\underline{\theta_i}}^{\overline{\theta_i}} \left(\theta_i\overline{v}_i(\theta_i) - U_i(\theta_i)\right)\phi_i(\theta_i)d\theta_i$$

subject to

(i) $\overline{v}_i(\cdot)$ is non-decreasing $\;\forall i\in N$

(ii)$y_{ij}(\theta)\in[0,1], \sum_{j=1}^{m} y_{ij}(\theta) \le 1, \sum_{i=1}^{n} y_{ij}(\theta) \le 1$ $\quad \forall i\in N,\ \forall j\in M,\ \forall\theta\in\Theta$

(iii) $U_i(\theta_i) = U_i(\underline{\theta_i}) + \int_{\underline{\theta_i}}^{\theta_i}\overline{v}_i(s)ds\ \ \forall i\in N,\ \forall\theta_i\in\Theta_i$

(iv) $U_i(\theta_i)\ge 0\ \forall i\in N,\ \forall\theta_i\in\Theta_i$

In the above formulation, the objective function is the total expected payment received by the search engine from all the advertisers. Note that constraints (iv) are the advertisers' interim individual rationality constraints while constraint (ii) is the feasibility constraint. Constraints (i) and (iii) are the necessary and sufficient conditions for the allocation and payment rule $f(\cdot) = (y_{ij}(\cdot), t_i(\cdot))_{i\in N, j\in M}$ to be Bayesian incentive compatible. These constraints are taken from [7]. We have a critical observation to make here. Note that in the above optimization problem, we have replaced the bid b_i by the actual type θ_i. This is because we are imposing the Bayesian incentive compatibility constraints on the allocation and payment rule and, hence, every advertiser will bid his true type. Thus, while dealing with the OPT mechanism, we can safely interchange θ_i and b_i for any $i\in N$. Define, as in [7], $J_i(\theta_i) = \theta_i - \frac{1-\Phi_i(\theta_i)}{\phi_i(\theta_i)}$ Then, following the same line of arguments as in [7], we can derive that

$$y_{ij}(\theta) = \begin{cases} 0 \quad \forall j=1,2,\ldots,m & : \ \text{if } J_i(\theta_i) < 0 \\ 1 \quad \forall j=1,2,\ldots,m<n & : \ \text{if } J_i(\theta_i) = J^{(j)} \\ 1 \quad \forall j=1,2,\ldots,n\le m & : \ \text{if } J_i(\theta_i) = J^{(j)} \\ 0 & : \ \text{otherwise} \end{cases} \qquad (2)$$

where $J^{(j)}$ is the j^{th} highest values among $J_i(\theta_i)$s. Now, recall the definition of $\overline{v}_i(\cdot)$. It is easy to write down the following expression:

$$\overline{v}_i(\theta_i) = E_{\theta_{-i}}[v_i(y(\theta_i,\theta_{-i}))] = E_{\theta_{-i}}\left[\sum_{j=1}^{m} y_{ij}(\theta_i,\theta_{-i})\alpha_j\right] \quad (3)$$

Now if we assume that $J_i(\cdot)$ is non-decreasing in θ_i, it is easy to see that the above solution $y_{ij}(\cdot)$, given by (2), will be non-decreasing in θ_i, which in turn implies, by looking at expression (3), that $\overline{v}_i(\cdot)$ is non-decreasing in θ_i. Thus, the solution to this relaxed problem actually satisfies constraint (i) under the assumption that $J_i(\cdot)$ is non-decreasing. Assuming that $J_i(\cdot)$ is non-decreasing, the solution given by

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(2) appears to be the solution of the optimal mechanism design problem for sponsored search auction. The condition that $J_i(\cdot)$ is non-decreasing in θ_i is met by most distribution functions such as Uniform and Exponential. In the rest of this paper, we will stick to the assumption that for every advertiser i, $J_i(\cdot)$ is non-decreasing in θ_i. We have interesting observation to make here.

Proposition 1 *If the advertisers are symmetric, that is* $\Theta_1 = \ldots = \Theta_n = \Theta$ *and* $\Phi_1(\cdot) = \ldots = \Phi_n(\cdot) = \Phi(\cdot)$ *and for every advertiser i, we have* $J_i(\cdot) > 0$ *and* $J_i(\cdot)$ *is non-decreasing, then for a given bid vector b, the OPT mechanism results in the same allocation as suggested by the GFP, the GSP, and the VCG mechanisms.*

3.2 Payment Rule

One can follow Myerson's line of attack to get the payment rule for the OPT mechanism. We are omitting the proof details here due to space constraints but may be found in [4]. The payment rule $p_i(\theta_i, \theta_{-i})$ for the OPT turns out be the following.

$$\begin{array}{ll} \frac{\alpha_\gamma}{\alpha_r} z_{i\gamma}(\theta_{-i}) + \frac{1}{\alpha_r}\sum_{j=r}^{\gamma-1} \beta_j z_{ij}(\theta_{-i}) & : \text{ if } 1 \le r \le (\gamma-1) \\ z_{i\gamma}(\theta_{-i}) & : \text{ if } r = \gamma \\ 0 & : \text{ otherwise} \end{array}$$

where $\gamma = m$ if $m < n$, otherwise $\gamma = n$. r is the position of advertiser i's Ad. $z_{ij}(.)$ are given as follows.

$$\begin{aligned} z_{i1}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_i(\theta_i) \ge J_{-i}^{(1)}\right\} \\ z_{i2}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(1)} > J_i(\theta_i) \ge J_{-i}^{(2)}\right\} \\ \vdots &= \vdots \\ z_{i\gamma}(\theta_{-i}) &= \inf\left\{\theta_i | J_i(\theta_i) > 0 \text{ and } J_{-i}^{(\gamma-1)} > J_i(\theta_i)\right\} \end{aligned}$$

3.3 The Case of Symmetric Advertisers

Proposition 1 shows that if the advertisers are symmetric, then the allocation rule under the OPT mechanism is the same as the GFP, the GSP, and the VCG mechanisms. The expression for the payment rule when the advertisers are symmetric can be computed easily. The reader is referred to [4] for the exact expressions. Now we state a very important result which shows that under some reasonable assumptions, the three mechanisms - OPT, VCG, and GSP, all fetch the same expected revenue to the search engine. The proof of this result is provided in [4].

Proposition 2 *(Revenue Equivalence of GSP, VCG, and OPT Mechanisms) Consider a sponsored search auction setting, in which the advertisers are risk neutral, the advertisers are symmetric, for each advertiser i, we have* $\phi_i(\cdot) > 0$*, the advertisers draw their types independently, and for each advertiser i, we have* $J_i(\cdot) > 0$ *and* $J_i(\cdot)$ *is non-decreasing function. For such a sponsored search auction environment, if* R_{GSP}, R_{VCG} *and* R_{OPT} *be the expected revenue earned by the search engine, against every search query received by the search engine, under the GSP, the VCG, and the OPT mechanisms, respectively, then*

$$\begin{array}{lll} R_{GSP} = R_{VCG} = R_{OPT} & : & \text{if } m < n \\ R_{VCG} \le R_{GSP} \le R_{OPT} & : & \text{if } n \le m \end{array}$$

4 Summary

In this paper, we formulated the sponsored search auction as a mechanism design problem in linear environment. Next, we proposed a new mechanism, called the OPT mechanism which improves upon the two most commonly used mechanisms GSP and VCG. We extended the classical revenue equivalence theorem to the setting of sponsored search auction and used it to show the revenue equivalence of the three mechanisms.

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