

# Optimal Auctions for Multi-Unit Procurement with Volume Discount Bids

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## Abstract

*Our attention is focused on designing an optimal procurement mechanism which a buyer can use for procuring multiple units of a homogeneous item based on bids submitted by autonomous, rational, and intelligent suppliers. We design elegant optimal procurement mechanisms for two different situations. In the first situation, each supplier specifies the maximum quantity that can be supplied together with a per unit price. For this situation, we design an optimal mechanism S-OPT (Optimal with Simple bids). In the more generalized case, each supplier specifies discounts based on the volume of supply. In this case, we design an optimal mechanism VD-OPT (Optimal with Volume Discount bids). The VD-OPT mechanism uses the S-OPT mechanism as a building block. The proposed mechanisms minimize the cost to the buyer, satisfying at the same time, (a) Bayesian incentive compatibility and (b) interim individual rationality.*

**Keywords:** Multi-unit procurement, incentive compatibility (IC), Bayesian incentive compatibility (BIC), individual rationality (IR), optimal mechanism.

## 1 Introduction

### 1.1 Motivation for the Work

A buyer, seeking to procure a high volume of a homogeneous item, would like to minimize the total cost of procuring the required number of units. In such a procurement scenario, the suppliers would compete with one another by offering volume discounts. In order to solve this problem in an optimal way, the buyer should know the true values of the costs of the suppliers. However, the suppliers may not be willing to provide this information in a truthful way. This is because, the suppliers are in general *autonomous, rational, and intelligent*. The suppliers might provide false bids

about their cost and capacity values in an attempt to increase their payoffs. With this as the setting, we attempt, in this paper, to design mechanisms which induce truth revelation by the suppliers. Such mechanisms are called *incentive compatible* mechanisms. There are two versions of *incentive compatibility*: (1) *dominant strategy incentive compatibility* (DSIC) (also called *strategy-proofness*) and (2) *Bayesian incentive compatibility* (BIC). A DSIC mechanism is one which makes it a best response for every supplier to reveal his true cost and capacity valuation, regardless of what the other suppliers reveal. A BIC mechanism is one which makes truth revelation a best response for a supplier, given that the other suppliers are truthful. The DSIC property is very strong and very desirable but difficult to achieve. On the other hand, the BIC property is much weaker and therefore easier to achieve.

In addition to the requirement of *incentive compatibility*, procurement mechanisms should be designed so as to induce the suppliers to participate voluntarily in the scheme. For this to happen, the scheme must ensure that the suppliers are not worse-off by participating than by not participating in the scheme. This property of an economic mechanism is called *individual rationality*. A buyer would like to minimize the cost of procurement under the conditions of *incentive compatibility* and *individual rationality*. A mechanism which achieves these properties is usually referred to as an *optimal mechanism*.

### 1.2 Contributions and Outline

In this paper, we consider the problem of designing an optimal multi-unit procurement auction that minimizes the cost to the buyer subject to (a) *Bayesian incentive compatibility* (which guarantees that truthful bidding is a best response for each supplier whenever all other suppliers also bid truthfully) and (b) *interim individual rationality* (which guarantees that the suppliers have non-negative payoffs if they participate in the auction). Two bidding scenarios are considered.

- In the first scenario, each supplier specifies the maximum quantity that he can supply and the per unit price. For this situation, we design an optimal mechanism which we call S-OPT (Optimal with Simple bids).
- In the more general second scenario, each supplier specifies discounts that depend on the volume of supply. In this case, we design an optimal mechanism which we call VD-OPT (Optimal with Volume Discount bids). The VD-OPT mechanism uses the S-OPT mechanism as a building block.

The rest of the paper is organized as follows. In Section 2, we present a review of relevant work to provide the context for the original contributions of this paper. In Section 3, we describe the physics of the problem and the model. We present the design of the S-OPT mechanism in Section 4 and the design of the VD-OPT mechanism in Section 5. Section 6 concludes the paper and presents avenues for further investigation.

## 2 A Review of Relevant Work

### 2.1 Multi-Unit Auctions with Volume Discount Bids

In a procurement context where a single buyer and multiple sellers who wish to exploit scale economies are present, a *volume discount auction* is appropriate. Here suppliers provide bids as a function of the quantity that is being purchased [4, 5]. The winner determination problem for this type of auction mechanism is to select a set of winning bids, where for each bid we select a price and quantity so that the total demand of the buyer is satisfied at minimum cost. The papers [4, 5], however, do not take into account rationality and strategic bidding by the suppliers.

Kameshwaran *et al* [6] have shown that single item, single attribute, multi-unit procurement with volume discount bids leads to a piecewise linear knapsack problem. In his work, Kameshwaran has developed several algorithms (exact, heuristic-based, and fully polynomial time approximation schemes) for solving such knapsack problems.

Kothari, Parkes, and Suri [7] consider single-item, single attribute, multi-unit procurement auctions where the bidders use marginal-decreasing, piecewise constant functions to bid for goods. The objective is to minimize the cost to the buyer. It is shown that the winner determination problem is a generalization of the classical 0/1 knapsack problem, and hence NP-hard. They have also addressed the computation of VCG payments. The authors provide a fully polynomial time approximation scheme for the generalized knapsack problem.

Dang and Jennings [3] consider multi-unit auctions where the bids are piecewise linear curves. Algorithms are

provided for solving the winner determination problem but these algorithms have exponential complexity in the number of bids.

In all the above scenarios, the focus is primarily on solving the winner determination problem with little or no emphasis on game theoretic properties such as incentive compatibility and individual rationality. In this paper, we attempt to design an optimal procurement auction which minimizes the cost while satisfying at the same time such game theoretic properties. We provide a review of relevant work in optimal auctions next.

### 2.2 Optimal Auctions

The problem of designing an optimal mechanism was first studied by Myerson [12]. Myerson considers the setting of a seller trying to sell a single object to one of several possible buyers and characterizes all auction mechanisms that are Bayesian incentive compatible and interim individually rational. From this, he derives the allocation rule and payment function for the optimal auction mechanism. This body of work is extended and also crucially used in our current paper.

Myerson's work can be easily extended to multi-unit auctions with unit demand. But problems arise when the unit-demand assumption is relaxed. We move into a setting of multi-dimensional type information which makes truth elicitation non-trivial. Several attempts have addressed this problem, albeit under some restrictive assumptions. It is assumed, for example, that even though the seller is selling multiple units (or even objects), the type information of the entities is still one dimensional [1, 13, 2].

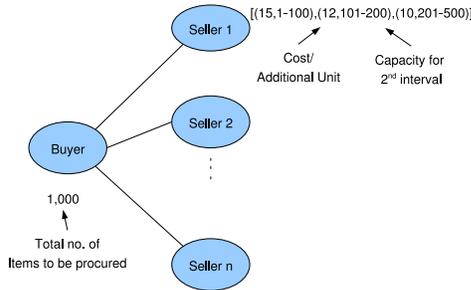
Manelli *et al* [11], provide a unique characterization of the optimal auction problem as a problem of finding an extreme point of a feasible set. They show for well-behaved distributions of the buyers' valuations that virtually any extreme point of the feasible set maximizes the seller's revenue. They also provide an algebraic procedure to check if a given mechanism is an extreme point of the feasible set.

Malakhov *et al* [9, 10] present a new approach to the problem of designing optimal auctions. They use a network flow approach and model the objective of designing an optimal auction to one of finding a shortest path on a lattice. The approach presented in [9, 10] is the only effort that handles the continuous case problem of optimal mechanism design in a discrete setting.

The model we adopt in our paper is somewhat based on this work. Kumar *et al* [8] consider a problem of optimal multi-unit procurement and characterize the optimal auction. They also devise a one-shot *get-your-bid* procurement auction for the model they devise.

### 3 The Model

There is a single buyer in the system who wants to purchase  $m$  homogeneous items from  $n$  suppliers, the set of suppliers being  $N = \{1, 2, \dots, n\}$ . See Figure 1. Let

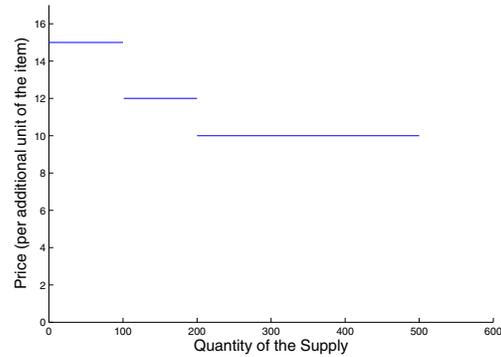


**Figure 1.** Procurement scenario for a single buyer

the maximum number of items that supplier  $i$  can supply be given by  $q_i$ . But not all these items may be supplied at the same price, because of *volume discounts*. Typically, the price per additional unit of the item for supply decreases as the number of items ordered from a supplier increases. See Figure 2. We assume the price per unit to be *piecewise constant*, which is common in practice. The piecewise constant character implies the pricing can be broken up into intervals for each of the suppliers. For the sake of convenience, we shall consider two cases:

1. *Simple Bids*: In this case, we assume that the private information of supplier  $i$  ( $i = 1, \dots, n$ ) is of the form  $(c_i, q_i)$  where  $q_i$  is the maximum number of units the supplier can provide and  $c_i$  is the per unit cost at which up to  $q_i$  units can be supplied.  $c_i$  could be interpreted as the production cost. The supplier may not always bid this true type information. A bid from supplier  $i$  would be of the form  $(\hat{c}_i, \hat{q}_i)$  where  $\hat{q}_i$  is what the supplier bids as his capacity and  $\hat{c}_i$  is the per unit cost at which he is prepared to sell these units.
2. *Volume Discount Bids*: In this case, each supplier will submit a *supply curve*, which is described in the following. Let  $I_i$  denote the set of price intervals for supplier  $i$  and  $q_{ij}$  be the capacity for the this supplier for the  $j^{\text{th}}$  ( $j \in I_i$ ) price interval; cost per unit item for this interval is denoted by  $c_{ij}$ . Let  $b_i \in B_i$  denote the true type that is  $b_i = \{c_{ij}, q_{ij}; j \in I_i\}$ , note that this is a set. At the time of submitting the bid, the supplier might want to submit  $\hat{b}_i = \{\hat{c}_{ij}, \hat{q}_{ij}; j \in \hat{I}_i\}$  reporting

the maximum number of items that he can supply to be  $\hat{m}_i$  instead of  $m_i$ .



**Figure 2.** Volume discount bid by supplier 1 (Example 2)

We provide two examples below to clarify the two situations.

**Example 1 (Simple Bids)**: Let the buyer be interested in procuring 1000 units of an item. Let there be 4 suppliers. Assume that the true type of supplier 1 is (10, 500). That is, supplier 1 can supply up to 500 units at a per unit cost of 10. Similarly, let the types of suppliers 2,3, and 4, respectively, be (8, 500), (12, 800), (6, 500), respectively. The actual bids of the suppliers could be different from their true types. For example, the bid of supplier could be (12, 400) rather than his true type (10, 500). In this paper, we are interested in designing a Bayesian incentive compatible and interim individually rational procurement mechanism that would minimize the cost to the buyer.

**Example 2 (Volume Discount Bids)**: Again, let the buyer be interested in procuring 1000 units of an item and let there be 4 suppliers. The bid structure here is more general than before. We consider the following four bids from the four suppliers.

Supplier 1	((15, 1 – 100), (12, 101 – 200), (10, 201 – 500))
Supplier 2	((12, 1 – 200), (10, 201 – 300), (8, 301 – 500))
Supplier 3	((16, 1 – 200), (14, 201 – 400), (12, 401 – 600), (10, 601 – 800))
Supplier 4	((10, 1 – 200), (6, 201 – 500))

The implication of the bid from supplier 1 (See Figure 2) is that the first 100 units would have a per unit cost of 15; the next 100 units would have a unit cost of 12; and the next 300 units would have a unit cost of 10. For example, if it is decided to buy 350 units from supplier 1, the total payment expected by supplier 1 is  $100 \times 15 + 100 \times 12 + 150 \times 10$ . As usual, the true types of the suppliers may be different from what they have bid.

## 4 S-OPT: Optimal Multi-Unit Procurement Auction with Simple Bids

### 4.1 Setup

Consider a single buyer who wants to buy  $m$  units of single item. Let the suppliers be from the set  $N = \{1, 2, \dots, n\}$ . Each supplier  $i$  has constant per unit production cost  $c_i \in [\underline{c}, \bar{c}] \subset [0, \infty)$  and also a maximum capacity  $q_i \in [q, \bar{q}] \subset [0, \infty)$ .  $F_i$  denotes the joint distribution of the per unit cost  $c_i$  and capacity  $q_i$ . We work with a joint distribution in order to allow for the cost and quantity values to be correlated. We assume that the suppliers are symmetric in the sense that they all have the same joint distribution function of the per unit cost and execution capacity, with the ranges over which  $c_i$  and  $q_i$  vary being the same for all suppliers say  $[\underline{c}, \bar{c}]$  and  $[q, \bar{q}]$ . Note that the actual values of  $(c_i, q_i)$  are known only to supplier  $i$ .

In our mechanism, the true type of each bidder (that is, supplier) is represented by  $b_i = (c_i, q_i)$  and each bid is represented by  $\hat{b}_i = (\hat{c}_i, \hat{q}_i)$ . The set of possible true types (and the bids) for supplier  $i$  is denoted by  $B_i$ . Note that in our setting  $B_i = [\underline{c}, \bar{c}] \times [q, \bar{q}]$ . Throughout this paper, we also adopt the notational convention of  $b_{-i}$  for  $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$ . Thus a type profile  $b$  is also represented as  $(b_i, b_{-i})$ .

Table 1 summarizes the notation that we use in respect of the model. In this section, we use the uppercase of a letter to denote the expectation of the component represented by the letter rather than the actual realization. Thus, for instance,  $X(\hat{b})$ , is an expectation of the value  $x(\hat{b})$ , which is the allocation vector based on the bid profile  $\hat{b}$ .

#### Some Definitions

We first define some notions that would be of use to us in the development of the *S-OPT* mechanism. The offered expected surplus gives us a measure of the surplus that is on offer to a supplier under a certain mechanism. That is, it characterizes the expected value of the profit to a supplier, were he to participate in the mechanism.

**Definition 1** For a procurement mechanism  $(x, t)$ , we define the offered expected surplus as:

$$\rho_i(\hat{c}_i, \hat{q}_i) = T_i(\hat{c}_i, \hat{q}_i) - \hat{c}_i X_i(\hat{c}_i, \hat{q}_i)$$

The offered expected surplus when the supplier  $i$  bids  $(\hat{c}_i, \hat{q}_i)$  is a measure of the expected transfer payment. The expected surplus  $\pi_i(\hat{c}_i, \hat{q}_i)$  of supplier  $i$  when the bid is  $(\hat{c}_i, \hat{q}_i)$  is given by

$$\begin{aligned} \pi_i(\hat{c}_i, \hat{q}_i) &= T_i(\hat{c}_i, \hat{q}_i) - c_i X_i(\hat{c}_i, \hat{q}_i) \\ &= \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i) X_i(\hat{c}_i, \hat{q}_i) \end{aligned}$$

**Table 1.** Notation used for the S-OPT mechanism

$m$	number of units sought to be procured by the buyer
$n$	number of suppliers
$N$	set of suppliers, $\{1, 2, \dots, n\}$
$i$	index for suppliers
$c_i$	actual production cost for supplier $i$
$q_i$	actual capacity of (maximum number of units that can be supplied by) supplier $i$
$b_i$	actual type of supplier $i$ , $b_i = (c_i, q_i)$
$\hat{c}_i$	reported unit production cost by supplier $i$
$\hat{q}_i$	reported capacity of supplier $i$
$\hat{b}_i$	reported type of supplier $i$ , $\hat{b}_i = (\hat{c}_i, \hat{q}_i)$
$B_i$	set of possible types of supplier $i$ $B_i = [\underline{c}, \bar{c}] \times [q, \bar{q}]$
$b$	a profile of types of the suppliers, represented by $(b_1, \dots, b_n)$
$\hat{b}$	a profile of bids of suppliers, represented by $(\hat{b}_1, \dots, \hat{b}_n)$
$B$	set of profiles of the types of the suppliers represented by $B_1 \times B_2 \times \dots \times B_n$
$b_{-i}$	a profile of types without supplier $i$ , represented by $(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$
$\hat{b}_{-i}$	a profile of types without supplier $i$ , represented by $(\hat{b}_1, \dots, \hat{b}_{i-1}, \hat{b}_{i+1}, \dots, \hat{b}_n)$
$x(\cdot)$	allocation rule of $f(\cdot)$
$t_i(\cdot)$	payment to a supplier $i$
$u_i(\cdot)$	utility of supplier $i$
$f_i(c_i q_i)$	probability of production cost to be $c_i$ , given the capacity is $q_i$
$F_i(c_i q_i)$	cumulative probability density on $f_i(c_i q_i)$
$H_i(c_i, q_i)$	virtual cost function when supplier $i$ has submitted the bid $(c_i, q_i)$
$\rho_i(\hat{c}_i, \hat{q}_i)$	expected surplus offered to $i$ when he is bidding $(\hat{c}_i, \hat{q}_i)$
$\pi_i(\hat{c}_i, \hat{q}_i)$	actual expected surplus for $i$ when he is bidding $(\hat{c}_i, \hat{q}_i)$

The true surplus  $\pi_i$  is equal to the offered surplus only if the mechanism is incentive compatible.

Following the idea from [12], we use a specific function called the *virtual cost function*, which is then used to rank the suppliers. The use of this virtual cost function in place of the actual cost bids, decreases the procurement cost possibly at the cost of efficiency.

**Definition 2** We define virtual cost as

$$H_i(c_i, q_i) = c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$$

This virtual cost parameter is similar to the virtual value parameter used in [12], except that here we allow for the cost and the quantity values to be correlated.

## Two Assumptions

The following assumptions are required. While these assumptions are not so strong as to limit the application of the results presented, they simplify the model enough to derive certain results which would not otherwise be achievable.

- **Assumption 1:** For all  $i = 1, 2, \dots, n$ , the joint density function  $f_i(c_i, q_i)$  is completely defined for all  $c_i \in [\underline{c}, \bar{c}]$  and  $q_i \in [\underline{q}, \bar{q}]$ , so that the conditional density function  $f_i(c_i | q_i)$  has full support.
- **Assumption 2:** For all  $i = 1, 2, \dots, n$ , the virtual cost function  $H_i(c_i, q_i)$  defined in definition 2 is non-decreasing in  $c_i$  and non-increasing in  $q_i$ . It is to be noted that this assumption imposes a condition on the relation between the cost and quantity of each supplier. This is similar to the regularity assumption in [12]. It holds especially when the cost and the quantity are independent of each other.

## 4.2 The S-OPT Mechanism

A procurement mechanism with quasi-linear utilities consists of:

1. an *allocation function*  $x : B \rightarrow \mathcal{R}_+^n$  which specifies the number of units to be allocated to each of the bidders, and
2. a *payment function*  $t : B \rightarrow \mathcal{R}^n$  which specifies the amount of money transferred from the buyer to the suppliers.

As auction designers, we are looking for the allocation function  $x$  and the transfer function  $t$  that minimizes the expected payment to be made by the buyer i.e.

$$\min \pi(x, t) = \mathbf{E}_b[\sum_{i=1}^n t_i(b)]$$

subject to:

1. *Feasibility:*  $x_i(b) \leq q_i \quad \forall i = 1, \dots, n$  and  $b \in B$
2. *Individual Rationality:* The expected interim payoff for each bidder is non-negative. That is,  $\pi_i(b_i) = \mathbf{E}_{b_{-i}}[t_i(b) - c_i x_i(b)] \geq 0$ .
3. *Bayesian Incentive Compatibility:* We require that truth revelation be optimal for each supplier, provided the other suppliers bid truthfully. That is,  $\mathbf{E}_{b_{-i}}[t_i(b_i, b_{-i}) - c_i x_i(b_i, b_{-i})] \geq \mathbf{E}_{b_{-i}}[t_i(\hat{b}_i, b_{-i}) - c_i x_i(\hat{b}_i, b_{-i})], \forall i \in N, \forall b_i \in B_i \forall \hat{b}_i \in B_i$ .

We receive a cost minimizing mechanism for the buyer as solution to this optimization problem. Any mechanism which satisfies the constraints given above and achieves a

minimum cost for the buyer is said to be an optimal auction on the lines of Myerson [12].

The problem of designing an optimal mechanism for the model we had discussed in Section 3 is different from the classic optimal auction presented in [12] in the sense that the demand is not of unit quantity. The buyer has a finite demand and allocates possibly multiple units to each supplier. It is this multidimensional type of the suppliers that make this an interesting and non-trivial problem. We not only have to consider the cost but also maximum quantity as private information. This is non-trivial in the sense that the suppliers are not of unit capacity type or un-capacitated. This bi-dimensional type profile of the suppliers makes the application of some traditional optimizing schemes unsuitable. This is illustrated in the example below.

## 4.3 An Illustrative Example

We show by means of an example that the problem S-OPT is non-trivial. Consider the Example 1 already given in Section 3. The buyer has a requirement for 1000 units. Suppose that there are four suppliers with  $(c_i, q_i)$  values of  $S_1 : (10, 500)$ ,  $S_2 : (8, 500)$ ,  $S_3 : (12, 800)$  and  $S_4 : (6, 500)$ . Now assume that the buyer were to conduct the classic  $K^{th}$  price auction, where the marginal payment to a supplier is equal to the cost of the first losing supplier.

The design of the uniform  $K^{th}$  price auction is such that it is a weakly dominant strategy for the suppliers to bid their true cost values. The auction may ensure that it is optimal for the suppliers to bid their true costs but does not deter them from possibly altering their quantity bids. To see this, consider that all suppliers bid truthfully both the cost and the quantity bids. The allocation that would happen would be  $S_1 : 0$ ,  $S_2 : 500$ ,  $S_3 : 0$ ,  $S_4 : 500$  so as to minimize the total payment. Under this allocation the payment to  $S_4$  would be  $10 \times 500 = 5000$  currency units. If he bids his quantity to be 490, the allocation changes to  $S_1 : 10$ ,  $S_2 : 500$ ,  $S_3 : 0$ ,  $S_4 : 490$  giving him a payment of  $12 \times 490 = 5880$  currency units. This clearly is not incentive compatible.

Thus it is quite evident that such uniform price mechanisms are not applicable to the case where both cost and quantity are private information. The intuitive explanation for this could be that by underbidding their capacity values, the suppliers create a fictitious shortage of resources in the system thereby forcing the buyer to pay overboard for use of the virtually limited resources.

## 4.4 Designing the S-OPT Mechanism

### 4.4.1 Characterizing the Optimal Solution

We now present an important lemma which characterizes the set of all incentive compatible and individually rational

mechanisms.

**Lemma 1** For the procurement problem under discussion, a feasible allocation rule  $x$  is Bayesian incentive compatible and individually rational if the expected allocation  $X_i(c_i, q_i)$  is non-increasing in cost valuation  $c_i$  for all  $q_i \in [\underline{q}, \bar{q}]$ , and the offered surplus  $\rho_i(\hat{c}_i, \hat{q}_i)$  is of the form

$$\rho_i(\hat{c}_i, \hat{q}_i) = \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(y, \hat{q}_i) dy \quad (1)$$

We also assume that  $\rho_i(\hat{c}_i, \hat{q}_i)$  must be non-negative and non-decreasing in  $\hat{q}_i$  for all  $\hat{c}_i \in [\underline{c}, \bar{c}]$ .

**Proof:** Suppose that  $X_i(c_i, q_i)$  is non-increasing in cost valuation  $c_i$  for all  $q_i$ . Also suppose that the offered expected surplus is of the form

$$\rho_i(\hat{c}_i, \hat{q}_i) = \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(y, \hat{q}_i) dy$$

such that  $\rho_i(\hat{c}_i, \hat{q}_i)$  is non-negative and non-decreasing in  $\hat{q}_i$  for all  $\hat{c}_i \in [\underline{c}, \bar{c}]$ . It can be easily seen that the suppliers do not benefit from overbidding their capacity valuation because they may end up not being able to complete the units allocated to them which may result in a penalty for them. Similarly, the suppliers being rational will not bid below their cost valuations as they will incur a loss by doing so. So, we can safely assume  $\hat{q}_i \leq q_i$ . This is a crucial observation for the rest of the proof.

For the  $\rho_i(\hat{c}_i, \hat{q}_i)$  function that we have chosen, the expected surplus of the supplier is,

$$\begin{aligned} \pi_i(\hat{c}_i, \hat{q}_i) &= \rho_i(\hat{c}_i, \hat{q}_i) + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i) \\ &= \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}} X_i(y, \hat{q}_i) dy \\ &\quad + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i) \\ &= \rho_i(\bar{c}, \hat{q}_i) + \int_{\hat{c}_i}^{c_i} X_i(y, \hat{q}_i) dy \\ &\quad + \int_{c_i}^{\bar{c}} X_i(y, \hat{q}_i) dy + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i) \\ &= \rho_i(\bar{c}, \hat{q}_i) + \int_{c_i}^{\bar{c}} X_i(y, \hat{q}_i) dy \\ &\quad + \int_{\hat{c}_i}^{c_i} X_i(y, \hat{q}_i) dy + (\hat{c}_i - c_i)X_i(\hat{c}_i, \hat{q}_i) \end{aligned}$$

Now, since  $X$  can never be negative and  $\hat{c}_i \geq c_i$ , we obtain,

$$\pi_i(\hat{c}_i, \hat{q}_i) \leq \rho_i(\bar{c}, \hat{q}_i) + \int_{c_i}^{\bar{c}} X_i(y, \hat{q}_i) dy$$

As per our assumption,  $\rho_i(\hat{c}_i, \hat{q}_i)$  is non-negative and non-decreasing in  $\hat{q}_i$  for all  $\hat{c}_i \in [\underline{c}, \bar{c}]$ . Using the fact that  $\hat{q}_i \leq q_i$  we have,

$$\begin{aligned} \pi_i(\hat{c}_i, \hat{q}_i) &\leq \rho_i(\bar{c}, \hat{q}_i) + \int_{c_i}^{\bar{c}} X_i(y, \hat{q}_i) dy \\ &= \rho_i(c_i, \hat{q}_i) \\ &\leq \rho_i(c_i, q_i) \\ &= \pi_i(c_i, q_i) \end{aligned}$$

The last equality follows from the fact that mechanism is Bayesian incentive compatible. We have shown that for all

$\hat{c}_i \in [\underline{c}, \bar{c}]$  and all  $\hat{q}_i \in [\underline{q}, \bar{q}]$ ,  $\pi_i(\hat{c}_i, \hat{q}_i) \leq \pi_i(c_i, q_i)$ . This proves that the mechanism is Bayesian incentive compatible under the conditions mentioned in the lemma. Since according to the conditions  $\rho_i(c_i, q_i)$  is non-negative, so is  $\pi_i(c_i, q_i)$ . This means that the mechanism that satisfies the conditions specified in Lemma 1 is individually rational as well.

## 4.5 The S-OPT Allocation and Payment Rules

In this section, we present the allocation rule and the payment function for the S-OPT mechanism. The first thing to note before we proceed with the design of the mechanism is that in order to minimize the cost of procurement of units, the mechanism should be such that the additional cost paid by the buyer as informational rent must be minimized as much as possible.

Here we present an allocation rule and a payment function which we claim is an optimal mechanism for the procurement problem. Before we proceed, recall the definition of the virtual cost function

$$H_i(c_i, q_i) = c_i + \frac{F_i(c_i|q_i)}{f_i(c_i|q_i)}$$

We would be using these  $H_i(c_i, q_i)$  values to compute the assignment vector. Now we rank the suppliers on the basis of their  $H_i(c_i, q_i)$  values and keep allocating units to them at their full capacity in increasing order of  $H_i$  values until all the units are allocated. Let  $[i]$  denote the supplier with  $i^{th}$  lowest  $H_i(c_i, q_i)$  value and  $[\bar{i}]$  be the supplier such that  $[\bar{i}]$  satisfies,

$$\begin{aligned} \sum_{j=[1]}^{[\bar{i}]-1} q_{[j]} &< m \text{ and} \\ \sum_{j=[1]}^{[\bar{i}]} q_{[j]} &\geq m \end{aligned}$$

Then the allocation function is given by

$$x_{[i]} = \begin{cases} \hat{q}_{[i]}, & [i] < [\bar{i}] \\ m - \sum_{j=[1]}^{[\bar{i}]-1} \hat{q}_{[j]}, & [i] = [\bar{i}] \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

With this allocation function, it is easy to see that  $X_i(c_i, q_i)$  is non-increasing in  $c_i$  and non-decreasing in  $q_i$ . Given this allocation rule, the payment function that would make this an optimal auction for the procurement problem is

$$t_i(\hat{b}) = c_i x_i(\hat{b}) + \int_{c_i}^{\bar{c}} X_i(y, \hat{q}_i) dy \quad (3)$$

## 5 VD-OPT: Optimal Multi-Unit Procurement Auction with Volume Discount Bids

The setup for this section remains the same as that of Section 3. The additional notation that have been used in this section has been summarized in Table 2. We will give an important result with regard to the allocation.

**Table 2.** Additional Notation for the VD-OPT mechanism

$q_i$	maximum capacity for supplier $i$
$q_{ij}$	true capacity of supplier $i$ for $j^{\text{th}}$ interval
$c_{ij}$	true unit production cost of supplier $i$ for $j^{\text{th}}$ interval
$fcc_i$	Averaged out cost for supplier $i$ at his full capacity which is same as $\frac{\sum_{j=1}^{I_i} c_{ij} * q_{ij}}{q_i}$
$FC$	set of suppliers who at a particular step of algorithm got allocated to their maximum capacity

**Proposition 1:** There always exists an allocation in the optimal auction such that all the suppliers with the possible exception of one, would be supplying to their maximum capacity.

**Proof:** We will show that for any allocation that is optimal, we can find an allocation which satisfies the above property or the allocation itself is not optimal. Let us say we have an optimal allocation, where there are  $k$  suppliers who are not supplying to the maximum of their capacity. Without loss of generality, we can assume that they are  $1, 2, \dots, k$ . Let their cost per unit for the last interval be  $c_{1j_1}, c_{2j_2}, \dots, c_{kj_k}$  respectively.

**Claim:**  $c_{1j_1} = c_{2j_2} = \dots = c_{kj_k}$ .

**Proof:** For contradiction, let us assume that there are some suppliers for which the above claim is not satisfied. Now we can find  $x, y \in 1, 2, \dots, k$  such that  $c_{xj_x} > c_{yj_y}$ . Since  $y$  is not allocated to its full capacity, we can unallocate a unit from  $x$  and allocate it to  $y$  to get a less procurement cost hence, the allocation is not optimal giving us the required contradiction.

To continue the proof of Proposition 1, we unassign units from 1 and assign it to 2. This can never give us a greater procurement cost as the bids follow volume discounts. We keep repeating this till 2 is allocated to his full capacity. We do similar reallocation from 1 to  $3, \dots, k$ . The allocation thus obtained leaves possible only one supplier namely 1 partially allocated and the rest are fully allocated. This proof is only existential and necessary to show the validity of the allocation method explained next.

In this section, we reduce our problem VD-OPT to S-OPT, that is, we shall be using S-OPT solver as a black box.

**The case of suppliers with large capacities (possibly infinite, that is, un-capacitated bidders)** Without loss of generality, we can assume that the bidders bid for at most  $m$  items. We can simply limit our view of the bids submitted by these bidders to  $m$  items and ignore the costs after  $m$  items. From now on, we shall consider only the case where bids have been placed for the supply of at most  $m$  items.

**Reduction of VD-OPT to S-OPT** For all the suppliers, calculate the average cost at full capacity as  $fcc_i = \frac{\sum_{j=1}^{I_i} c_{ij} * q_{ij}}{q_i}$ . Additionally, S-OPT needs probability density function  $f_i(fcc_i|q_i)$  and cumulative distribution function  $F_i(fcc_i|q_i)$ . This must be found by convoluting the costs over probability density function. This essentially means that probability density and cumulative distribution must be found at cost  $fcc_i$ , taking into consideration the joint probability distribution of cost and quantity that exists up to the point of  $fcc_i$ . We can now use S-OPT solver to rank the suppliers and allocate them to their full extent. The suppliers who have been allocated to their maximum capacity can no longer take part in the auction; let us call them  $FC$ . We must now calculate payments for the suppliers of  $FC$  by the S-OPT solver scheme and drop them from the auction process. There could be a possible supplier who has not been allocated to his full extent, say he has been allocated some  $m'$  units. We deviate from S-OPT solver and do not allocate him any units at all. We recurse on  $m'$  units of the item and rest of the bidders that are left in the auction procedure.

**Observation 1:** If the total number of units of items of all the supplier's bids before running the above procedure was at least  $m$  then, even after running the procedure also, it will be at least  $m$ . This happens because at every recursive step we preserve at least  $m'$  units of the item. To see why this happens, consider the supplier who was the candidate for partial allocation by the S-OPT solver. Then that supplier even in the next step would be capable of supplying  $m'$  units. So we have at least one supplier who would supply us the required  $m'$  units of item.

**Proposition 2:** If the total number of units in the bids of all the suppliers is at least  $m$ , then our algorithm would always find an allocation such that all the  $m$  units are procured.

**Proof:** This follows directly from Observation 1 and the fact that if  $m'$  units were chosen for partial allocation, then there is at least one supplier who can supply these  $m'$  units of the item.

**Proposition 3:** The allocation and payments are Bayesian incentive compatible and interim individually rational.

**Proof:** Our procedure inherits these properties from the S-OPT solver. If the VD-OPT solver has done non-optimal allocation, then it must have done it at some step of the re-

cursor. Then this implies that at that step, the S-OPT solver did non-optimal allocations, which is a contradiction. So our procedure ensures IR and IC.

**Proposition 4:** The allocation is optimal.

**Claim:** If all the supplier of set  $FC$  and the rest  $m'$  units were optimally allocated then the auction is optimal.

**Proof:** At any recursive step of the algorithm, it is not feasible to allocate any more units to the suppliers of the  $FC$  set. So the only possible transfer that can be thought of is from a supplier in  $FC$  to a supplier that is not part of the  $FC$  set. Now for contradiction, let us assume that there is some unit that could be transferred from bidder  $i \in FC$  to some other supplier  $j \notin FC$  saving some money. Now, there are two possible cases: either  $j$  is fully allocated or it is not fully allocated. If  $j$  is fully allocated then, it should have been picked up by the algorithm in preference to  $i$  in this very recursive step giving us a less cost, implying a contradiction. Now consider the other case where  $j$  is not fully allocated and so is  $i$  after the transfer, but this is in contradiction with Proposition 1. Hence, the claim holds.

**Proof of Proposition 4:** This follows directly from the above claim and the recursion that takes place on  $m'$  items.

**Proposition 5:** The procedure invokes S-OPT solver  $n$  times in the worst case.

**Proof:** In every step of recursion, at least one supplier is allocated to his full capacity and is removed from the auction as part of the  $FC$  set. So by the end of  $n$  steps, the auction has to terminate.

## 6 Future Work

There are several natural extensions to this work. One possible extension would be a case where different units of the item have some attribute associated with them that affect the purchase decision for the item. Another possible extension could be in the direction of combinatorial auctions. In the case of combinatorial auctions, the assumption for discount is that, for bid by each of the supplier, the cost of a set of items is no more than the sum of costs of individual items. Another possible avenue for research could be to look for cost-minimizing auctions under volume discount settings that satisfy some additional properties such as allocative efficiency.

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