Price of Anarchy of Network Routing Games with Incomplete Information

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Abstract. We consider a class of networks where n agents need to send their traffic from a given source to a given destination over m identical, non-intersecting, and parallel links. For such networks, our interest is in computing the worst case loss in social welfare when a distributed routing scheme is used instead of a centralized one. For this, we use a noncooperative game model with price of anarchy as the index of comparison. Previous work in this area makes the complete information assumption, that is, every agent knows deterministically the amount of traffic injected by every other agent. Our work relaxes this by assuming that the amount of traffic each agent wishes to send is known to the agent itself but not to the rest of the agents; each agent has a belief about the traffic loads of all other agents, expressed in terms of a probability distribution. In this paper, we first set up a model for such network situations; the model is a noncooperative Bayesian game with incomplete information. We study the resulting games using the solution concept of *Bayesian Nash equilibrium* and a representation called the *type agent representation*. We derive an upper bound on price of anarchy for these games, assuming the total expected delay experienced by all the agents as the social cost. It turns out that these bounds are independent of the belief probability distributions of the agents. This fact, in particular, implies that the same bounds must hold for the complete information case, which is vindicated by the existing results in the literature for complete information routing games.

1 Introduction

The motivation for this paper comes from several recent papers (for example, [6], [10], [7], [2], [1]), where the authors use the index *price of anarchy* [9] to measure the worst case loss of network performance when switching from a centralized routing scheme to distributed one. This happens due to selfish behavior of non-cooperative network users when routing of the traffic is done in a distributed fashion. An important implicit assumption made by the authors in all

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these articles, while analyzing the underlying network game model, is that all the users have *complete information* about the game, including information such as how much traffic is being sent through the network by the other users. This assumption severely restricts the applicability of the model to the real world traffic networks. Very few papers in the current literature have explored the model under the incomplete information assumption. A recent paper by Martin Gairing et al [3] considers network routing games with incomplete information which is very similar to ours but they are concerned with developing a polynomial time algorithm for computing a pure Bayesian Nash equilibrium of the underlying game. [3] also gives bounds for the coordination ratio which is similar to our result on price of anarchy. In our paper, the focus is on incompleteness of information available to the users (we also use the synonym agents for users in the rest of the paper). We relax the assumption of complete information by saying that agents do not deterministically know the loads being injected by the other agents, however, having observed the traffic pattern on the network over a sufficiently long time, each agent has a belief (that is, a probability distribution) about the loads of the other agents. In this more realistic scenario, the underlying network routing game becomes a *Bayesian game of incomplete information*. It would be now interesting to study how *incompleteness* of information would affect the performance of routing schemes.

Our main results in this paper show that the bounds on price of anarchy for incomplete information routing games are independent of the belief probability distributions of the agents. This fact, in particular, implies that the same bounds must hold for complete information routing games since they are the special case of incomplete information games. In fact, the bounds we derive match with the known bounds for complete information routing games, thus validating our results.

1.1 Contributions and Outline

In this paper, our main objective is to analyze network routing games with incomplete information. We consider a special class of traffic networks where n agents need to send their traffic from a given source to a given destination over m identical, non-intersecting, and parallel links. To analyze the games that result when we relax the complete information assumption. The sequence in which we progress is as follows.

- *The Model: Bayesian Routing Game* (Section 2): We first develop a Bayesian game model of the routing game with incomplete information. Next, we present an equivalent game of complete information using the *type agent representation* [8].
- *Analysis of the Model: Bayesian Nash Equilibrium* (Section 3): We work with the type agent representation of the Bayesian routing game developed above and characterize the Bayesian-Nash equilibria of the game.
- *Bounds on Price of Anarchy* (Section 4): Next, we define the price of anarchy for Bayesian routing games and compute an upper bound for it. The bound computed turns out to be independent of the belief probability distributions of the agents.

To the best of our knowledge, this is the first time game theoretic analysis and price of anarchy are being investigated in the context of routing games with incomplete information. Our definition of social cost is *total delay experienced by all the agents*, where the delay of an individual agent is the total traffic being assigned to the link on which the agent is transmitting. This definition for social cost is the same as that employed by Roughgarden [10] and Lucking *et al* [7], but different from that employed by Koutsoupias and Papadimitriou [9].

2 Bayesian Routing Games

Consider a network in which there are m identical, non-intersecting, parallel links, $\{L^1, L^2, \ldots, L^m\}$, to carry the traffic from source S to destination D. There are n users (agents), denoted by the set $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, who need to send traffic from S to D. We assume that the traffic injected by an agent cannot be split across the links. We also assume that each agent A_i can inject any amount of traffic load from a given finite set $\mathcal{W}_i = \{1, 2, \ldots, K\}$. We view the traffic load as an abstract quantity, however, depending upon the context, it can be measured in terms of appropriate units such as Kbps or Mbps. We shall use symbol w_i to denote the actual traffic load generated by A_i. We would like to mention here that our model is static in the sense that w_i denotes the average traffic load of the agent A_i. We assume that before the show starts, the load w_i is the private information of agent A_i and is unknown to the rest of the agents. Sticking to standard phraseology used in the context of incomplete information games (Bayesian games) [8], we prefer to call w_i as the type of agent A_i and hence $\mathcal{W}_i$ becomes the set of all possible types of agent A_i. We use the symbol $\mathcal{W} = \mathcal{W}_1 \times \mathcal{W}_2 \ldots \times \mathcal{W}_n$ to denote the set of type profiles of the agents and $\mathcal{W}_{-i} = \mathcal{W}_1 \times \ldots \times \mathcal{W}_{i-1} \times \mathcal{W}_{i+1} \ldots \times \mathcal{W}_n$ to denote the set of type profiles of agents excluding A_i. We use the symbol w and w_{-i}, respectively to denote an element of the sets $\mathcal{W}$ and $\mathcal{W}_{-i}$, respectively. $\triangle\mathcal{W}$ is the set all probability distributions over $\mathcal{W}$ and similarly $\triangle\mathcal{W}_{-i}$ is the set of all probability distributions over $\mathcal{W}_{-i}$. We also assume that each agent A_i has a belief function $p_i : \mathcal{W}_i \mapsto \triangle\mathcal{W}_{-i}$. That is, for any possible type w_i of the agent A_i, the belief function specifies the probability distribution over the set $\mathcal{W}_{-i}$, representing what agent A_i would believe about the other agents' type if its own type were w_i. The beliefs $(p_i)_{\mathcal{A}}$ are said to be *consistent* [8] iff there is some common prior distribution $P \in \triangle\mathcal{W}$ over the set of type profiles w such that each agent's belief given its type is just the conditional probability distribution that can be computed from the prior distribution by Bayes formula in following manner.

$$p_i(w_{-i}\,|w_i\,) = \frac{P(w_{-i}, w_i)}{\sum\limits_{s_{-i} \in \mathcal{W}_{-i}} P(s_{-i}, w_i)};\ \forall A_i \in \mathcal{A}$$

In this paper, we will stick to this assumption of consistent beliefs. Under this assumption, the probability that agent A_i's type is w_i can be given by $t_i(w_i) = \sum_{w_{-i} \in \mathcal{W}_{-i}} P(w_{-i}, w_i)$. Now consider the following problem with regard to this network. The agents are rational, selfish, and non-cooperative, and are left free to route their traffic through the network. Each agent, knowing the fact that the other agents are also doing likewise, *independently* tries to compute a strategy for routing its traffic that yields minimum expected delay. The decision problem of each agent is to choose the best link for sending its traffic through the network. We define the set of all the links as a pure strategy set of any agent A_i and denote it by $\mathcal{L}_i$. We shall be using symbol l_i to denote a particular pure strategy of agent A_i. We also define a mixed strategy for agent A_i as any valid probability distribution over the set of pure strategies. We use the symbol $\mathcal{T}_i$ to denote the set of all the mixed strategies of agent A_i, and the symbol τ_i to denote a particular mixed strategy of agent A_i, that is, $\mathcal{T}_i = \triangle\mathcal{L}_i$, and $\tau_i \in \mathcal{T}_i$. We use the symbol $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \ldots \times \mathcal{L}_n$ to denote the set of pure strategy profiles of the agents and $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2 \ldots \times \mathcal{T}_n$ to denote the set of mixed strategy profiles of the agents. We can define $\mathcal{L}_{-i}$ and $\mathcal{T}_{-i}$ in a similar way as earlier. We use the corresponding lowercase letters to denote individual elements of the above sets.

The payoff to an agents here is the delay experienced by the agent, which is defined as the total traffic passing through the link on which its own traffic is running. Thus, we define $u_i : \mathcal{L} \times \mathcal{W} \mapsto \Re$ as a payoff function for agent A_i, where $u_i(l, w) = \sum_{j:l_j = l_i} w_j$ is the payoff to agent A_i under the strategy profile l and traffic profile w and $u_i(\tau, w) = \sum_{l \in \mathcal{L}} \left(\prod_{A_i \in \mathcal{A}} \tau_i(l_i) \right) u_i(l, w)$ is the payoff to agent A_i under the strategy profile τ and traffic profile w. This model describes the following Bayesian game

$$\Gamma^b = \left\{(\mathcal{A}), (\mathcal{L}_i)_{A_i \in \mathcal{A}}, (\mathcal{W}_i)_{A_i \in \mathcal{A}}, (p_i)_{A_i \in \mathcal{A}}, (u_i)_{A_i \in \mathcal{A}}\right\} \tag{1}$$

Harsanyi [5], proposed a game, called the *Selten Game* to represent such games in normal form of an augmented complete information game. Myerson [8] calls such a representation *type-agent representation.*

2.1 Type Agent Representation for Bayesian Routing Games

In type-agent representation, it is assumed that there is one virtual player (or agent) for every possible type of every player (or agent) in the given Bayesian game and thus the set of agents gets augmented. In order to differentiate these agents from the actual agents (i.e. network users), we prefer to call them as *traffic agents.* Thus, for the Bayesian game Γ^b given by (1), the set of agents in the type-agent representation becomes $\mathcal{A}^t = \bigcup_{i \in \mathcal{A}} \mathcal{A}_i^t$, where $\mathcal{A}_i^t = \{A_{i1}, A_{i2}, \ldots, A_{iK}\}$ represents the set of traffic agents for agent A_i.

The pure strategy and mixed strategy sets of traffic agent A_{ij} are the same as the pure and mixed strategies set of agent A_i. That is, $\mathcal{L}_{ij} = \mathcal{L}_i$ and $\mathcal{T}_{ij} = \mathcal{T}_i$. We shall use symbols l_{ij} and τ_{ij} to denote a particular pure and mixed strategy respectively for the traffic agent A_{ij}. We use the symbol $\mathcal{L}^t = \mathcal{L}_{11} \times \mathcal{L}_{12} \ldots \times \mathcal{L}_{nK}$ to denote the set of pure strategy profiles of traffic agents and $\mathcal{T}^t = \mathcal{T}_{11} \times \mathcal{T}_{12} \times \ldots \times \mathcal{T}_{nK}$ to denote the set of mixed strategy profiles of traffic agents. Once again, $\mathcal{L}^t_{-ij}$, $\mathcal{L}^t_{-i}$, $\mathcal{T}^t_{-ij}$, and $\mathcal{T}^t_{-i}$ have their usual interpretations. Also, we shall use lowercase letters to denote individual elements of the above sets. Two other quantities that are of use are $(l^t|w)$ and $(\tau^t|w)$. The first one represents the pure strategy profile of the agents for a given pure strategy profile of the traffic agents and a given type profile of the agents. The second quantity is a mixed strategy counterpart of the first one. That is, $(l^t|w) = (l_{1w_1}, l_{2w_2}, \ldots l_{nw_n})$ and $(\tau^t|w) = (\tau_{1w_1}, \tau_{1w_2}, \ldots \tau_{nw_n})$, where $w = (w_1, \ldots, w_n)$. In the type agent representation, the payoff to any traffic agent A_{ij} is defined to be the conditionally expected payoff to agent A_i in Γ^b given that j is A_i's actual type. Formally, for any agent A_i in $\mathcal{A}$ and any type w_i, the payoff function $v_{iw_i} : \mathcal{L}^t \mapsto \Re$ in the type-agent representation is defined in the following way.

$$v_{iw_i}(l^t) = \sum_{w_{-i} \in \mathcal{W}_{-i}} p_i\left(w_{-i}|w_i\right) u_i\left((l^t|w), (w_{-i}, w_i)\right) \tag{2}$$

where $w = (w_1, \ldots, w_i, \ldots, w_n) = (w_{-i}, w_i)$. Similarly, for the mixed strategy case, the payoff is given by the following equation.

$$v_{iw_i}(\tau^t) = \sum_{l^t \in \mathcal{L}^t} \left(\prod_{A_{pq} \in \mathcal{A}^t} \tau_{pq}(l_{pq}) \right) v_{iw_i}(l^t) \tag{3}$$

Substituting the value of equation (2) in equation (3) leads to the following alternative form of $v_{iw_i}(\tau^t)$:

$$v_{iw_i}(\tau^t) = \sum_{w_{-i} \in \mathcal{W}_{-i}} p_i\left(w_{-i}|w_i\right) u_i\left(\tau^t|w, w\right) \tag{4}$$

With these definitions, the type-agent representation

$$\Gamma = \left\{ (\mathcal{A}^t), (\mathcal{L}_{ij})_{A_{ij} \in \mathcal{A}^t}, (v_{ij}(.))_{A_{ij} \in \mathcal{A}^t} \right\}$$

is indeed a complete information game in strategic form and may be viewed as a representation of the given Bayesian game.

2.2 Payoffs to Agents

Before moving to the next section, we would like to define an important quantity, namely, payoff to agent A_i in the incomplete information game. The following relations give the expected payoff to agent A_i when the pure strategy and mixed strategy profile of the traffic agents are l^t and τ^t, respectively.

$$u_i(l^t) = \sum_{w \in \mathcal{W}} P(w) u_i(l^t|w, w)$$

$$u_i(\tau^t) = \sum_{l^t \in \mathcal{L}^t} \left(\prod_{A_{pq} \in \mathcal{A}^t} \tau_{pq}(l_{pq}) \right) u_i(l^t) = \sum_{w \in \mathcal{W}} P(w) u_i(\tau^t|w, w) \tag{5}$$

The second expression for $u_i(\tau^t)$ in equation (5) can be obtained by substituting the value of $u_i(l^t)$ in the first expression of $u_i(\tau^t)$. Also, by making use of equations (2) and (4), it is very simple to get the following alternative expressions for $u_i(l^t)$ and $u_i(\tau^t)$.

$$u_i(l^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(l^t); \quad u_i(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(\tau^t) \tag{6}$$

This completes the definition of the type agent representation for Bayesian routing games.

3 Nash Equilibria for Bayesian Routing Games

We now carry out a game theoretic analysis of the Bayesian routing game using the solution concept of Bayesian Nash equilibrium. First we consider the Bayesian game Γ^b with a fixed type profile of the agent, that is $\mathcal{W}_i = \{w_i\}$ is a singleton set. In this situation, the game reduces to a game of complete information because the Bayesian form Γ^b and type agent representation Γ are essentially the same as $\mathcal{W}_i$ and p_i are now redundant and no more required in the Bayesian form Γ^b. Let $w = (w_1, w_2, \ldots, w_n)$ be a given traffic profile of the agents. If $\tau = (\tau_1, \ldots, \tau_i, \ldots, \tau_n) = (\tau_i, \tau_{-i})$ is a mixed strategy Nash equilibrium for this game, it can be shown that [4]:

$$u_i(\tau, w) = M^j(\tau, w) + w_i \left(1 - \tau_i(L^j)\right) \text{ if } L^j \in S_i$$
$$u_i(\tau, w) \leq M^j(\tau, w) + w_i \left(1 - \tau_i(L^j)\right) \text{ if } L^j \notin S_i$$

where $M^j(\tau, w)$ is the expected traffic arising on link L^j if all the agents send their traffic according to the strategy profile τ. Summing up the $u_i(\tau, w)$ values over all the m links and making use of the fact that $\sum_j M^j(\tau, w) = \sum_i w_i$, it is easy to get the following upper bound on expected payoff of agent A_i in the case of Nash equilibrium.

$$u_i(\tau, w) \leq \frac{1}{m} \left\{ (m-1)w_i + \sum_i w_i \right\} \tag{7}$$

Let Γ have a Nash equilibrium τ^t with support $S^t = S_{11} \times S_{12} \times \ldots \times S_{nK}$, where $S_{ij} \subset \mathcal{L}_{ij}$ (see [8] for more details on support), then for each traffic agent A_{iw_i}, there must exist η_{iw_i} such that

$$\eta_{iw_i} = v_{iw_i}((L^j, \tau^t_{-iw_i}))\, \forall L^j \in S_i w_i; \;\; \eta_{iw_i} \leq v_{iw_i}((L^j, \tau^t_{-iw_i}))\, \forall L^j \notin S_i w_i \tag{8}$$

Let Γ have a Nash equilibrium τ^t with support $S^t = S_{11} \times S_{12} \times \ldots \times S_{nK}$, where $S_{ij} \subset \mathcal{L}_{ij}$, then for each traffic agent A_{iw_i}, there must exist η_{iw_i} such that

$$\eta_{iw_i} = v_{iw_i}((L^j, \tau^t_{-iw_i}))\, \forall L^j \in S_i w_i; \; \eta_{iw_i} \leq v_{iw_i}((L^j, \tau^t_{-iw_i}))\, \forall L^j \notin S_i w_i \tag{9}$$

$$\sum_{L^j \in S_{iw_i}} \tau_{iw_i}(L^j) = 1 \tag{10}$$

$$\tau_{iw_i}(L^j) = 0\, \forall L^j \notin S_{iw_i} \; ; \tau_{iw_i}(L^j) \geq 0\, \forall L^j \in S_{iw_i} \tag{11}$$

It is easy to show that η_{iw_i} that we obtain by solving the above system of equations is indeed the expected payoff of traffic agent A_{iw_i} under τ^t. Further, we can derive the following expression for the expected payoff of the agent A_i under Bayesian Nash equilibrium τ^t.

$$\sum_{w_i \in \mathcal{W}_i} t_i(w_i) u_i(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) v_{iw_i}(\tau^t) = \sum_{w_i \in \mathcal{W}_i} t_i(w_i) \eta_{iw_i}$$

The above relation reduces to the following relations (see [4] for details):

If $L^j \in S_{iw_i} \; \forall w_i \in \mathcal{W}_i$ then

$$u_i(\tau^t) = \sum_{w_i} t_i(w_i) \left\{ w_i + \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \right\}$$

otherwise

$$u_i(\tau^t) \leq \sum_{w_i} t_i(w_i) \left\{ w_i + \sum_{w_{-i}} p_i(w_{-i}|w_i) M^j((\tau^t|w)_{-i}, w_{-i}) \right\}$$

where $M^j(.)$ has its usual interpretation. We can also derive the following alternative form of the above expressions (see [4]for details):

If $L^j \in S_{iw_i} \; \forall w_i \in \mathcal{W}_i$ then

$$u_i(\tau^t) = \sum_{w_i} t_i(w_i) w_i + \sum_{w} P(w) \left\{ M^j((\tau^t|w), w) - \tau_{iw_i}(L^j) w_i \right\} \tag{12}$$

Otherwise

$$u_i(\tau^t) \leq \sum_{w_i} t_i(w_i) w_i + \sum_{w} P(w) \left\{ M^j((\tau^t|w), w) - \tau_{iw_i}(L^j) w_i \right\} \tag{13}$$

If we sum up $u_i(\tau^t)$ over all the links and make use of the above relations then it is easy to get the following bound:

$$u_i(\tau^t) \leq \frac{1}{m} \left\{ (m-1) \sum_{w_i} t_i(w_i) w_i + \sum_{w} P(w) \sum_{i} w_i \right\} \tag{14}$$

Note the similarity between the above expression and expression (7).

4 Price of Anarchy of Bayesian Routing Games

In this section, we develop a notion of price of anarchy for Bayesian routing games and derive an upper bound for it. For the sake of convenience and self-sufficiency, we discuss the case of complete information routing games.

4.1 Price of Anarchy for Complete Information Routing Games

Consider the complete information routing game $\Gamma^c = \{\mathcal{A}, (\mathcal{L}_i)_{\mathcal{A}}, (u_i(.))_{\mathcal{A}}\}$. For this game, we define the following quantities.

$$S(\tau) = \text{Social cost under mixed strategy profile } \tau = \sum_i u_i(\tau, w)$$
$$\underline{S} = \text{Optimal social cost} = \min_{\tau} S(\tau)$$
$$\underline{S}^* = \text{Social cost under the best Nash equilibrium} = \min_{\tau:\tau \text{ is a NE}} \{S(\tau)\}$$
$$\overline{S^*} = \text{Social cost under the worst Nash equilibrium} = \max_{\tau:\tau \text{ is a NE}} \{S(\tau)\}$$
$$\overline{S} = \max_{\tau} S(\tau)$$

The following inequality is a trivial consequence of the above definitions.

$$\underline{S} \leq \underline{S}^* \leq \overline{S^*} \leq \overline{S}$$

It is straightforward to see that under the centralized routing scheme, the central authority would route the agents' traffic in way that yields the social cost $\underline{S}$. However, under a distributed routing scheme, the agents would like to route their traffic in a way suggested by a Nash equilibrium strategy profile. Therefore, the worst possible scenario that may arise out of distributed routing scheme is to have social cost equal to $\overline{S^*}$. Thus, the following ratio ϕ, called as *price of anarchy for the game* Γ^c, measures the worst case loss in social welfare because of switching from centralized routing scheme to distributed routing scheme.

$$\phi = \frac{\overline{S^*}}{\underline{S}} = \frac{\text{Social cost under the worst Nash equilibrium}}{\text{Optimal social cost}}$$

In what follows we compute an upper bound for the above ratio. By making use of the upper bound for $u_i(\tau, w)$, given in (7), it is easy to show that

$$\text{If } \; m = 1 \; \text{ then } \; \overline{S^*} = \underline{S} = n \sum_i w_i \; \text{ else } \; \overline{S^*} \leq \frac{n + (m-1)}{m} \sum_i w_i \; ; \; \underline{S} \geq \sum_i w_i$$

The above relations result in the following theorem about bounds on price of anarchy for complete information routing games:

Theorem 1. For a complete information routing game with m identical parallel links and n users, the price of anarchy ϕ can be bounded in the following way:

$$\text{If } \; m = 1 \; \text{ then } \; \phi = 1 \quad \text{else } \; 1 \leq \phi \leq \left\{ \frac{n + (m-1)}{m} \right\}$$

Remarks:

- These bounds are not tight for the case when $1 < m$. However, for a given value of m, one can find a better approximation of $\underline{S}$ and get a tighter bound.
- Another interesting case is when $m = n$. For this case, it is easy to see that $1 \leq \phi \leq 2 - \frac{1}{m}$.
- For $m = (n-1)$, $1 \leq \phi \leq 2$.

4.2 Price of Anarchy for Bayesian Routing Games

Now we consider the Bayesian game Γ^b and define the price of anarchy for it. With regard to the type-agent representation Γ of the Bayesian game Γ^b, we define $S(\tau^t) = \sum_i u_i(\tau^t)$ to be the social cost under mixed strategy profile τ^t. The quantities $\underline{S}, \underline{S}^*, \overline{S^*}$, and $\overline{S}$ have their usual meaning. The following inequality is a direct consequence of the above definitions:

$$\underline{S} \leq \underline{S}^* \leq \overline{S^*} \leq \overline{S}$$

The price of anarchy ψ for the Bayesian game Γ^b is defined by the ratio:

$$\psi = \frac{\overline{S^*}}{\underline{S}} = \frac{\text{Social cost under the worst BN equilibrium}}{\text{Optimal social cost}}$$

In what follows, we compute an upper bound on ψ. By making use of the upper bound for $u_i(\tau^t)$, given in (14), it is easy to show that

$$\text{If } m = 1 \text{ then } \overline{S^*} = \underline{S} = n \sum_w P(w) \sum_i w_i \text{ , otherwise}$$

$$\overline{S^*} \leq \frac{(m-1)\sum\limits_{i=1}^{n} \sum\limits_{w_i \in \mathcal{W}_i} t_i(w_i) w_i + n \sum\limits_w P(w) \sum\limits_i w_i}{m} \text{ ; } \underline{S} \geq \sum_i^n \sum_{w_i \in \mathcal{W}_i} t_i(w_i) w_i$$

We have made use of equation (12) to bound $\underline{S}$ from below when $1 < m$. The above relation results in the following bounds on price of anarchy for incomplete information routing games.

$$\text{If } m = 1 \text{ then } \psi = 1, \text{otherwise } 1 \leq \psi \leq \frac{1}{m} \left\{ (m-1) + n \frac{\sum\limits_w P(w) \sum\limits_i w_i}{\sum\limits_i^n \sum\limits_{w_i \in \mathcal{W}_i} t_i(w_i) w_i} \right\}$$

A little algebra shows that $\sum\limits_w P(w) \sum\limits_i w_i = \sum\limits_i^n \sum\limits_{w_i \in \mathcal{W}_i} t_i(w_i) w_i$. The following theorem captures the bounds on the price of anarchy for the incomplete information case.

Theorem 2. For an incomplete information routing game with m identical parallel links and n users, the price of anarchy ψ is bounded by

$$\text{If } \ m = 1 \ \text{ then } \ \psi = 1 \quad \text{else } \ 1 \leq \psi \leq \left\{ \frac{n + (m-1)}{m} \right\}$$

Remarks:

– Note that the bounds on price of anarchy are the independent of the belief probability distributions p_i of the agents and hence the same bounds hold for complete information case as well.

– These bounds are not tight for the case when $1 < m$. However, for a given value of m, one can find a better approximation of $\underline{S}$ and get a tighter bound.
– For $m = n$, $1 \leq \psi \leq 2 - \frac{1}{m}$
– For $m = (n - 1)$, $1 \leq \psi \leq 2$.

5 Conclusions

In this paper, we have extended game theoretic analysis to network routing games with incomplete information. We have derived an upper bound on price of anarchy for such games. The results show that the bound is independent of the belief probability distributions of the agents. We believe our work is a good first step in answering the following more general question: how does distributed routing affect (improve or degrade) the actual performance when agents have probabilistic rather than deterministic information about other agents? Further investigation of this question will lead to several interesting directions for future work: (1) tighter bounds on price of anarchy; (2) price of anarchy with other metrics such as average delay, throughput, etc. (3) more general routing situations; and (4) mechanism design for evolving better routing protocols for such networks.

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