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A Nash Bargaining Approach to Retention Enhancing Bid Optimization in Sponsored Search Auctions With Discrete Bids

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Abstract— **Bid optimization is now becoming quite popular in sponsored search auctions on the web. Given a keyword and the maximum willingness to pay of each advertiser interested in the keyword, the bid optimizer generates a profile of bids for the advertisers with the objective of maximizing customer retention without compromising the revenue of the search engine. In this paper, we present a bid optimization algorithm that is based on a Nash bargaining model where the first player is the search engine and the second player is a virtual agent representing all the bidders. We make the realistic assumption that each bidder specifies a maximum willingness to pay values and a discrete, finite set of bid values. We show that the Nash bargaining solution for this problem always lies on a certain edge of the convex hull such that one end point of the edge is the vector of maximum willingness to pay of all the bidders. We show that the other endpoint of this edge can be computed as a solution of a linear programming problem. We also show how the solution can be transformed to a bid profile of the advertisers.**

Index Terms— **Bid optimizers, Nash bargaining, advertiser retention, sponsored search auctions, internet advertising, mechanism design**

I. Background and Motivation

The on-line advertising market is growing at a fast pace, from over \$40 billion in 2007 to nearly \$80 billion by 2010. The prevalent form of on-line advertising constitutes displaying of Ads against the search results by the search engine. A majority of these search engine companies sell their advertising space through auctions which are popularly known as *sponsored search auctions (SSA)*, also called *Ad auctions, position auctions*, and *keyword auctions*. Battelle [3] provides more details on the history of search engines and the auction models used by them for selling the Ads. We foresee two major potential threats to any search engine company's business in the current scenario:

- *Emergence of alternate search engines*: Today, due to exponential growth in the space of on-line advertising, the number of search engine companies is increasing steadily and more interestingly, a majority of these search engine companies have a comparable market share. In the face of such a competition, each search engine company faces the threat of its advertisers[1] switching to an alternative search engine in quest of better bang-per-buck. Such a switching can severely affect the revenue of the search engine company.
- *Emergence of unscrupulous service providers*: Imagine that an unscrupulous service provider emerges and starts offering an on-line service X where any bidder for a given search engine can report his *willingness to pay* for a keyword. After receiving such bids from all the bidders, the service X performs some computation and confidentially recommends a bid value to each of the registered bidders. Now the bidders go and submit these bids in a sponsored search auction. These bid values can possibly hurt the auctioneer's revenue and in fact, can even lead to zero revenue to the auctioneer. An example of such a bid profile is discussed by Kannan, Garg, Subbian and Narahari in [12]. This is equivalent to having a virtual mediator that facilitates collusion among unknown bidders, leading to a decrease in the auctioneer revenue by eliminating competition among bidders.

Given the above threats, the search engine companies need to focus on ways of responding to these threats in a proactive manner. Using a *Bid Optimizer* in conjunction with sponsored search auctions is a recent phenomenon which tries to address the first threat mentioned above. The bid optimizer helps a search engine company improve the retention of its advertisers by giving them better bang-per-buck.[2] A bid optimizer is a software agent that automatically chooses bid values on behalf of the bidders. Bidders will provide a target budget to exhaust within a given number of days and maximum bid value (willingness to pay). The bid optimizer aims to maximize the bang-per-buck by adjusting the bid amount based on the projected keyword traffic and remaining budget.

It is interesting to note that a sponsored search auction without any bid optimizer is more favorable to the auctioneer than to the bidders and this results in the tendency of the bidders switching to an alternative search engine company for displaying their Ads. The bid optimizer is an attempt towards reducing such a bias in the system. The objective of a typical bid optimizer is to strike a balance between the reduction in the revenue of the auctioneer (search engine company) versus increase the retention of bidders. Such an objective is achieved by providing enhanced utilities to the bidders, without significantly compromising the revenue to

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[1]In this paper, we use the terms - *"auctioneer"* with *search engine company* and *"bidders"* with *advertisers* interchangeably.

[2]Number of clicks the bidder's website receives for \$1 spent to display the advertisement. For example, if the website receives 80 clicks for \$100 spent, bang-per-buck is 0.8

the auctioneer.

The second threat of *unscrupulous service provider* has received almost no attention in the industry or the academia. In this paper, our objective is to address this threat also, in addition to the first threat. Towards this end, we present a novel bid optimization algorithm which is based on a two person Nash bargaining model where the auctioneer is the first player and a virtual aggregated agent representing all the n bidders is the second player. The solution to the Nash bargaining problem yields a fair division of utility between the auctioneer and bidders satisfying the axioms of the Nash bargaining model. The algorithm takes the maximum willingness to pay for a keyword of all individual bidders and recommends a bid profile for the bid optimizer to bid on behalf of the bidders. Our solution can be seen as associating the bid optimizer to a keyword rather than bidders as is done by the existing bid optimizers.

A. Relevant Work

The motivation for our work comes from several recent articles on sponsored search auctions. We present here the works that are most relevant to our paper.

Edelman, Ostrovsky and Schwarz [6] investigate the generalized second price auction mechanism which is widely used to sell sponsored slots. The papers by Iyengar and Kumar [11] and by Garg, Narahari, and Reddy [9] have proposed optimal auctions under various assumptions on the click through rates to maximize revenue to the auctioneer. Goodman [10] has explored a simple method for selling advertising, that is immune to both click fraud and impression fraud called pay-per-percentage of impressions. Meek, Maxwell, and Chickering [13] describe a family of stochastic auctions that are incentive compatible and describe situations where stochastic auctions are useful. Aggarwal, Goel, and Motwani [1] present a truthful auction under separable click-through rates and have proved the revenue-equivalence between the generalized second price (GSP) auction and the proposed truthful auction.

The idea of a bid optimizer has been a subject of interest for the researcher in the recent times. Kitts and Leblanc [4] present a trading agent for sponsored search auctions. The trading agent computes how much to bid given the bids of the other advertisers. Ashlagi, Monderer, and Tennenholtz [2] suggest a mediator based scheme which implements the Vickrey-Clark-Groves (VCG) outcome for different classes of sponsored search auction. Borgs *et al* [5] propose a modified class of mechanisms with small random perturbations to avoid cycling among bidders who attempt to optimize their utility by equalizing their return-on-investment across all keywords. Budget optimization is the problem of bidders, on deciding bids for a set of interested keywords so as to maximize bang-per-buck with in the given budget. The contribution from Feldman, Muthukrishnan, Pal, and Stein [7] analyze the complexities of solving such budget optimization problems.

Recently, Kannan, Garg, Subbian, and Narahari [12] have proposed a novel approach, based on a Nash bargaining model, to bid optimization. Their approach takes as input a keyword and a profile of maximum willingness to pay values of the bidders for that keyword and generates a profile of bids for the bidders based on a Nash bargaining model. In the Nash bargaining model, the first player is the auctioneer and the second player is a virtual agent representing all the bidders. The profile of bids produced by their algorithm turns out to be a locally envy free equilibrium. Their experimentation shows that the bid profile generated produces somewhat less revenue to the auctioneer but produces increased payoffs to the bidders thus enhancing the probability of customer retention.

A critical assumption in the above paper [12] is that the bid of each bidder is a continuous variable that belongs to a real interval. A majority of the commercial websites offering bid optimization typically expect the advertisers to provide a discrete set of bids from which the bid optimizer selects a suitable bid value. The approach presented in [12] cannot be used in the discrete setting since the the entire approach is crucially based on the assumption that the bids are continuous. Our current paper develops the approach for the discrete setting. It turns out that the modifications required to handle the discrete setting are quite non-trivial and this is the main contribution of this paper.

B. Contributions and Outline of the paper

We have a sponsored auction setting in which given a keyword, there is a set of advertisers or bidders interested in that keyword. Each bidder specifies a maximum willingness to pay and a set of possible bid values. The problem facing the auctioneer is to select a profile of bid values to maximize customer retention without compromising the revenue of the auctioneer. To solve this problem, we proceed in the following way in this paper.

- In Section 2, we set up a Nash bargaining model with the auctioneer as the first player and a virtual agent representing all the bidders as the second player. This formulation is different from what is presented in [12] in terms of the allowed bid values which are now discrete.
- In Section 3, we prove certain critical properties of the Nash bargaining solution for the formulated problem, which help us to determine the solution. We next show how the Nash bargaining solution can be mapped to a profile of bids.
- Section 4 concludes the paper and provides several directions for future work.

II. Nash Bargaining Based Bid Optimizer

A typical bid optimizer is focused on an individual bidder and it acts as an advocate for the individual bidder by means of selecting a bid value on behalf of the bidder. However, in our solution approach, instead of making the bid optimizer select a bid value for each individual bidder separately, we make it select a bid profile for all bidders such that it provides enhanced utilities to the bidders without compromising much the revenue of auctioneer. This bid profile is selected based on a Nash bargaining solution of

Notation	Explanation
A	Auctioneer
B	Bid Optimizer
n	Number of bidders
m	Number of slots. Assumption is $m < n$
N	Set of bidders $\{1, 2, \cdots, n\}$
M	Set of position/slots $\{1, 2, \cdots, m\}$
i	Bidder index
j	Slot index
$\overline{s}_i \in \Re^+$	Maximum willingness to pay by bidder i. $\overline{s}_i \geq 1$
S_i	Finite Discrete Bid set of bidder i, $\{s_1, s_2, \cdots, \overline{s}_i\}$ This can be system generated between 0 and $\overline{s}_i$ or can be specified by the bidder
S	Set of all bid profiles $S_1 \times S_2 \times \ldots \times S_n$
s	A bid profile $s = (s_1, s_2, \cdots, s_n) \in S$
$\overline{s}$	Bid profile when each bidder bids $\overline{s}_i$, $i = 1, 2, \cdots, n$
$p_i(s)$	Payment of bidder i on bid profile s
$u_i(s)$	Utility of bidder i on bid profile s
$y_{ij}(s) \in \{0, 1\}$	Allocation of bidder i to slot j for profile s.
$U_A(s)$	Utility of auctioneer for profile s
$U_B(s)$	Utility of bid optimizer for profile s. $U_B(s) = \sum_{i=1}^{n} u_i(s)$
β_j	Probability that any bidder's ad gets clicked when displayed on slot j. Also known as Click Through Rate(CTR). Assumptions are (1)β_j are independent of bidders (2) $\beta_j \geq \beta_{j+1}$, $\forall j = 1, 2, \cdots, m-1$
$\overline{U}_A$	$\max_{s \in S} \; U_A(s)$
$\overline{U}_B$	$\sum_{j=1}^{m} \beta_j \overline{s}_j$, where bidders are indexed in the decreasing order of $\overline{s}_i$

TABLE I
NOTATION

the bargaining problem between the auctioneer and a virtual aggregated bidder representing all the n bidders. Our solution approach takes all bidders' maximum willingness to pay for a keyword as input and recommends a bid profile.

A. Mechanism Design Formulation for SSA with Bid Optimizer

In this section, we discuss the components of mechanism design [8] for the Nash Bargaining Based Bid Optimizer discussed in this paper.

- Players : Bidders $N = \{1, 2, \cdots, n\}$
- An outcome in the case of sponsored search auction with bid optimizer may be represented by a vector $x = (y_{ij}, p_i)_{i \in N, j \in M}$, where y_{ij} represents whether bidder i is allocated to the slot j and p_i denotes the price-per-click charged from the bidder i. The set of feasible alternatives is then

$$X = \left\{ (y_{ij}, p_i)_{i \in N, j \in M} \;\middle|\; \begin{array}{l} y_{ij} \in [0,1] \, \forall i \in N, \forall j \in M, \\ \sum_{i=1}^{n} y_{ij} \leq 1 \, \forall j \in M, \\ \sum_{j=1}^{m} y_{ij} \leq 1 \, \forall i \in N, \\ p_i \geq 0 \, \forall i \in N \end{array} \right.$$

- The utility function[3] of bidder i is given, for $x = (y_{ij}, p_i)_{i \in N, j \in M}$, by $u_i(x, s_i) = (\sum_{j=1}^{m} y_{ij}\beta_j)(\overline{s}_i - p_i)$
- Allocation and Payment Rule $f(\cdot)$: The general structure of the allocation and payment rule for this case is $f(s) = (y_{ij}(s), p_i(s))_{i \in N, j \in M}$ where $s = (s_1, \ldots, s_n)$ is a bid vector of the bidder. The functions $y_{ij}(\cdot)$ form the allocation rule and the functions $p_i(\cdot)$ form the payment rule. We use the allocation and the payment rule explained in Preliminaries section of [12].

B. Nash Bargaining Formulation

In this section, we define a bargaining problem between the auctioneer and an aggregated bidder representing all the n bidders and the auctioneer. For this purpose, we assume that auctioneer uses GSP as the underlying auction mechanism. The bargaining space is defined in two dimensional Cartesian space, with utility of auctioneer $U_A(s)$ along the x-axis and the utility of aggregated bidder $U_B(s) = \sum_{i=1}^{n} u_i(s)$ along the y-axis. The Nash bargaining solution needs a convex hull F and disagreement point v to be defined on this utility space.

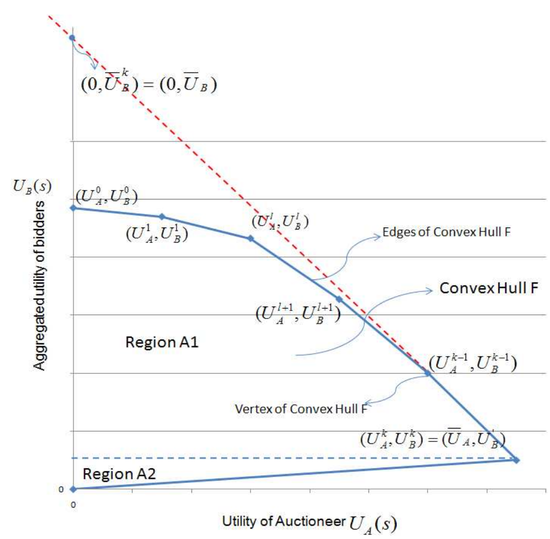

Fig. 1. Structure of the Solution Space

Let us run the GSP mechanism for all possible bid profiles $s \in S$. The auctioneer's strategies are *Play* and *Don't Play*. If the auctioneer decides not to play then the utilities of the aggregated bidder and the auctioneer turns out to be (0,0). If the auctioneer chooses to play, then the utility of the auctioneer is U_A and the utility of the aggregated bidder is U_B. The convex hull F for the utility space $(U_A(s), U_B(s)) \, \forall s \in S$, is shown in Figure 1. The point (0,0), where the auctioneer does not choose to play, becomes the disagreement point v by invoking the rational threats criterion [14].

In the next section, we prove certain critical properties of the Nash Bargaining Solution for the bargaining problem formulated in this section.

[3]The utility function is as perceived by the Bid Optimizer. The $\overline{s}_i$ elicited by the bidder i can be different from the bidder's true value

III. NASH BARGAINING SOLUTION FOR BID OPTIMIZER

In order to explain our ideas, we proceed as follows:

- We analyze the possible structure of the convex hull for the bargaining problem between the auctioneer and the aggregated bidder.
- Next, we show how to compute the solution of such a Nash bargaining problem
- Finally, we show how to translate back this Nash bargaining solution into meaningful terms such as bids of the bidders.

Let us consider $(U_A^0, U_B^0), (U_A^1, U_B^1), \cdots, (U_A^k, U_B^k)$ are vertices of the convex hull F in region $A1$ (see Figure 1). Each of these vertices maps to at least one bid profile $s \in S$ and hence interchangeably sometimes used as $(U_A(s), U_B(s))$. The vertices are indexed clockwise on Quadrant I of the Cartesian Coordinate with variable l. Let equation of the edge formed out of the vertices $(U_A^{l-1}, U_B^{l-1}), (U_A^l, U_B^l)$ be $U_B = -m_l U_A + \overline{U}_B^l$, where m_l is the slope of the edge and $\overline{U}_B^l$ is the y-intercept of the edge, such that $U_A^{l-1} \leq U_A \leq U_A^l$ and $U_B^{l-1} \geq U_B \geq U_B^l$.

A. Structure of the Convex Hull F

Lemma 1: U_B and U_A for any bid profile $s \in S$ satisfy the relation $U_A + U_B \leq \overline{U}_B$.

Proof: The proof for this Lemma, is provided in Lemma 5 of [12]. Hence, the vertex (U_A^k, U_B^k) in the Figure 1 is the same point $(\overline{U}_A, U_B^{'})$ for the bid profile $\overline{s}$. ■

Lemma 2: The bid profile $s = \overline{s} = (\overline{s}_1, \overline{s}_2, \cdots, \overline{s}_n)$ yields the utility pair $(\overline{U}_A, U_B^{'})$, where

$$U_B^{'} = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) (\overline{s}_i - \overline{s}_{i+1}).$$

Proof: The proof for this Lemma, is provided in Lemma 1 of [12]. ■

The vertex corresponding to the utility pair $(\overline{U}_A, U_B^{'})$ is also represented as (U_A^k, U_B^k).

Lemma 3: The NBS solution (U_A^*, U_B^*) cannot lie in the region $A2$ of the convex hull F shown in Figure 1.

Proof: From the figure it is evident that in the region $A2$, U_B decreases as U_A decreases. Let us assume a point $U_A - \delta < U_A^k$. The corresponding U_B be $U_B - \gamma < U_B^K$. Hence,

$$\begin{aligned} &(U_A - \delta)(U_B - \gamma) \\ &\leq U_A^k (U_B - \gamma) \\ &\leq U_A^k U_B^k \end{aligned}$$

■

Claim 1: In the region A1, for the edges $U_B = -m_l U_A + \overline{U}_B^l$, $\forall l \leq k-1$, $m_{l-1} < m_l < m_{l+1}$, $U_A^{l-1} < U_A^l < U_A^{l+1}$, $U_B^{l-1} > U_B^l > U_B^{l+1}$ and $\overline{U}_B^{l-1} < \overline{U}_B^l < \overline{U}_B^{l+1}$

Proof: As the edges $U_B = -m_l U_A + \overline{U}_B^l$, of the convex hull F are indexed clockwise in Quadrant I of the Cartesian coordinate with variable l, as described in Figure 1, the above relations hold good trivially. ■

As part of the next theorem, we prove that there exists an edge on the convex hull F with the maximum slope 1 and the maximum y-intercept $\overline{U}_B$.

Theorem 1: The line $U_A + U_B = \overline{U}_B$ is an edge of the convex hull F

Proof: We know that,

$$U_A^k = \overline{U}_A = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) \overline{s}_{i+1}$$

$$U_B^k = U_B^{'} = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) (\overline{s}_i - \overline{s}_{i+1})$$

Thus the point (U_A^k, U_B^k) satisfies the equation $U_A + U_B = \overline{U}_B$.

From Lemma 1, we have shown that $U_A(s) + U_B(s) \leq \overline{U}_B$. Hence, if we find another bid profile $s \in S$, such that $U_A(s) + U_B(s) = \overline{U}_B$, proves that the convex hull F, has an edge $U_A(s) + U_B(s) = \overline{U}_B$ formed out of point (U_A^k, U_B^k) and $(U_A(s), U_B(s))$.

Let us assume, the bidders i are indexed in decreasing order of their maximum willingness to pay $\overline{s}_i$. The bid profile $(s_1, 1, 0, \cdots, 0)$, where $s_1 > 1$ if $\overline{s}_1 > \overline{s}_2$, else $s_1 = 1$, also satisfies the equation $U_A(s) + U_B(s) = \overline{U}_B$. This implies that the vertices (U_A^k, U_B^k) and $(U_A(s_1, 1, 0, \cdots, 0), U_B(s_1, 1, 0, \cdots, 0))$, lie on the edge $U_A(s) + U_B(s) = \overline{U}_B$. ■

B. Computation of the Nash Bargaining Solution (NBS)

In this section, we find the NBS (U_A^*, U_B^*) for the bargaining problem between aggregated bidder and the auctioneer.

We know that for a given line $y = -mx + c$, the maximum value of the product xy is attained at the point $(\frac{c}{2m}, \frac{c}{2})$. Let us consider a portion of this line $y = -mx + c$, lying between the points (x_1, y_1) and (x_2, y_2), where $x_1 < x_2$. Under this consideration, the point $(\frac{c}{2m}, \frac{c}{2})$ can belong to following three cases.

- Case 1 : The point lies in between the start and the end points (x_1, y_1) and (x_2, y_2). That is, $y_1 > \frac{c}{2m}$ and $y_2 < \frac{c}{2m}$
- Case 2 : The point lies before the start point (x_1, y_1). That is, $y_1 \leq \frac{c}{2m}$
- Case 3 : The point lies after the end point (x_2, y_2). That is $y_2 \geq \frac{c}{2m}$

We formally define the function $maxxy$ based on this intuition.

Definition 1: Let $maxxy$ be the function that gives the maximum value of the product $U_A U_B$ of the edge formed out of point (U_A^{l-1}, U_B^{l-1}) and (U_A^l, U_B^l).

$$maxxy\left((U_A^{l-1}, U_B^{l-1}), (U_A^l, U_B^l)\right) = \begin{cases} \left(\frac{\overline{U}_B^l}{2m_l}, \frac{\overline{U}_B^l}{2}\right) \text{ if } \mathrm{U_B^l} < \frac{\overline{\mathrm{U}}_\mathrm{B}^\mathrm{l}}{2\mathrm{m_l}} \text{ and } \mathrm{U_B^{l-1}} > \frac{\overline{\mathrm{U}}_\mathrm{B}^\mathrm{l}}{2\mathrm{m_l}} \text{ (case 1)} \\ (U_A^{l-1}, U_B^{l-1}) \text{ if } \mathrm{U_B^{l-1}} \leq \frac{\overline{\mathrm{U}}_\mathrm{B}^\mathrm{l}}{2\mathrm{m_l}} \text{ (case 2)} \\ (U_A^l, U_B^l) \text{ if } \mathrm{U_B^l} \geq \frac{\overline{\mathrm{U}}_\mathrm{B}^\mathrm{l}}{2\mathrm{m_l}} \text{ (case 3)} \end{cases}$$

After defining the $maxxy$, for any given line segment between any two given points, we explore the existence of the three cases on all the edges $U_B = -m_l U_A + \overline{U}_B^l$ in the region $A1$ of our convex hull F.

Lemma 4: Case 2 of the function $maxxy$ never exists for the edge $U_B = -m_l U_A + \overline{U}^l_B$ of the convex hull F formed out of vertices (U_A^{l-1}, U_B^{l-1}) and (U_A^l, U_B^l) $\forall l \leq k$.

Proof: We know from Claim 1 that, $U_B^{l-1} > U_B^l > U_B^{l+1}$ and $\overline{U}_B^{l-1} < \overline{U}_B^l < \overline{U}_B^{l+1}$. Hence if we show that, even for the farthest edge $U_A + U_B = \overline{U}_B$ from y-axis, Case 2 of $maxxy$ does not exist, then for all edges $U_B = -m_l U_A + \overline{U}_B^l$ $0 < l < k$, the inequality $U_B^l \geq \frac{\overline{U}_B^l}{2}$ hold good. We know from Theorem 1 that for the bid profile $s = (s_1, 1, 0, \cdots, 0)$, where $s_1 > 1$ if $\overline{s}_1 > \overline{s}_2$, else $s_1 = 1$, lies on the edge $U_A + U_B = \overline{U}_B$. For this bid profile,

$$U_B(s) = \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) (\overline{s}_i - p_i) \right]$$

$$\begin{aligned} &= \beta_1 \overline{s}_1 + \beta_2 \overline{s}_2 + \ldots + \beta_{\mathrm{m}} \overline{s}_m - \beta_1 \\ &\geq \left(\tfrac{\beta_1 \overline{s}_1 + \beta_2 \overline{s}_2 + \ldots + \beta_{\mathrm{m}} \overline{s}_m}{2} \right) - \beta_1 \\ &\geq \tfrac{\overline{U}_B}{2} \end{aligned}$$

The last inequality is because of the fact that $\overline{U}_B \gg \beta_1$. ■

Lemma 5: Case 1 of the function $maxxy$ never exists for the edge $U_B = -m_l U_A + \overline{U}_B^l$ of the convex hull F formed out of vertices (U_A^{l-1}, U_B^{l-1}) and (U_A^l, U_B^l) $\forall l \leq k$.

Proof: To prove this Lemma, we have to show that $U_B^{l-1} > \frac{\overline{U}_B^{l-1}}{2}$ $\forall l \leq k$. We know from Claim 1 that, $\overline{U}_B^{l-1} < \overline{U}_B^l < \overline{U}_B^{l+1}$. Let us assume $\overline{U}_B^{l-1} + \delta = \overline{U}_B^l$.

Also, we know from Lemma 4 that,

$$\begin{aligned} U_B^{l-1} &\geq \tfrac{\overline{U}_B^l}{2} \\ &\geq \tfrac{\overline{U}_B^{l-1} + \delta}{2} \\ &\geq \tfrac{\overline{U}_B^{l-1}}{2} + \tfrac{\delta}{2} \\ &> \tfrac{\overline{U}_B^{l-1}}{2} \end{aligned}$$

The last inequality is because of the fact that $\delta \neq 0$. ■

Theorem 2: The NBS solution (U_A^*, U_B^*) for the convex hull F, always lie on the edge $U_B + U_A = \overline{U}_B$ that is the edge formed out of vertices (U_A^{k-1}, U_B^{k-1}) and (U_A^k, U_B^k).

Proof: We know from Lemma 4 and Lemma 5 that, the edges $U_B = -m_l U_A + \overline{U}_B^l$ $\forall l < k$, have only Case 3 of function $maxxy$ only as feasible solution and the farthest edge $U_B = -m_k U_A + \overline{U}_B^k$, has both Case 1 and Case 3 of $maxxy$ as feasible solution.

Hence, to show that NBS solution always lies on $U_B + U_A = \overline{U}_B$, we need to prove two cases. Case A, where all the edges satisfy Case 3 of $maxxy$. That is., $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < U_A^k U_B^k$. Similarly, Case B, where the farthest edge $U_A + U_B = \overline{U}_B$ satisfies Case 1 of $maxxy$ and all other edges satisfy Case 3 of $maxxy$. That is, $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < \frac{\overline{U}_B^2}{4}$.

<u>Case A</u> : $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < U_A^k U_B^k$

If we show that, $U_A^{l-1} U_B^{l-1} < U_A^l U_B^l$, $\forall l < k$, without loss of generality, we can say $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < U_A^k U_B^k$.

$$\begin{aligned} 0 &< U_A^{l-1} U_B^{l-1} - U_A^l U_B^l \\ &= U_A^{l-1}(-m_l U_A^{l-1} + \overline{U}_B^l) - U_A^l U_B^l \\ &= -m_l (U_A^{l-1})^2 + U_A^{l-1} \overline{U}_B^l - U_A^l(-m_l U_A^l + \overline{U}_B^l) \\ &= m_l[(U_A^l)^2 - (U_A^{l-1})^2] - \overline{U}_B^l (U_A^l - U_A^{l-1}) \\ &= m_l[(U_A^l)^2 - (U_A^{l-1})^2] - (m_l U_A^l + U_B^l)(U_A^l - U_A^{l-1}) \end{aligned}$$

The last equation follows because $U_B^l = -m_l U_A^l + \overline{U}_B^l$. Also we know that, $m_l = \frac{U_B^l - U_B^{l-1}}{U_A^l - U_A^{l-1}}$ in the above equation we get,

$$\begin{aligned} &= (U_B^l - U_B^{l-1})(U_A^l + U_A^{l-1}) - U_A^l (U_B^l - U_B^{l-1}) \\ &\qquad - U_B^l (U_A^l - U_A^{l-1}) \\ &= U_A^{l-1}(U_B^l - U_B^{l-1}) - U_B^l (U_A^l - U_A^{l-1}) \\ &= 2 U_A^{l-1} U_B^l - (U_A^l U_B^l + U_A^{l-1} U_B^{l-1}) \end{aligned}$$

We know from Claim 1 that $U_A^{l-1} U_B^l < U_A^l U_B^l$ and $U_A^{l-1} U_B^l < U_A^{l-1} U_B^{l-1}$. Hence the last equation holds good.

<u>Case B</u> : $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < \frac{\overline{U}_B^2}{4}$

We know that the maximum value of the product xy of the line of the $y = -mx + c$ is at the point $\left(\frac{c}{2m}, \frac{c}{2}\right)$. We know from Theorem 1 that, the slope m for line $U_B = -m_k U_A + \overline{U}_B^k$ is 1 and $c = \overline{U}_B$. Hence the maximum value of product $U_A^k U_B^k$ is attained at point $\left(\frac{\overline{U}_B}{2}, \frac{\overline{U}_B}{2}\right)$ and it lies in between the start and the end points (U_A^{k-1}, U_B^{k-1}) and (U_A^k, U_B^k). Thus we can say that, in this case $U_A^k U_B^k < \frac{\overline{U}_B^2}{4}$. We know from the above Case A that, $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < U_A^k U_B^k$ and we have shown in this case that, $U_A^k U_B^k < \frac{\overline{U}_B^2}{4}$. Hence, $U_A^1 U_B^1 < U_A^2 U_B^2 < \cdots < \frac{\overline{U}_B^2}{4}$. ■

Theorem 3: The NBS solution (U_A^*, U_B^*) for the convex hull F with disagreement point $(0, 0)$, is

$$(U_A^*, U_B^*) = \begin{cases} (U_A^k, U_B^k) = (\overline{U}_A, U_B^{'}) \text{ if } \mathrm{U_A^k} \leq \frac{\mathrm{\overline{U}_B}}{2} \\ \left(\frac{\overline{U}_B}{2}, \frac{\overline{U}_B}{2}\right) \text{ if } \mathrm{U_A^k} > \frac{\mathrm{\overline{U}_B}}{2} \end{cases}$$

Proof: We know from Theorem 2 that, the NBS solution for the convex hull F with disagreement point $(0, 0)$ lies on the edge $U_B = -m_k U_A + \overline{U}_B^k$. Also we know from Lemma 4 and 5 that, the edge $U_B = -m_k U_A + \overline{U}_B^k$ can have the maximum value of the product $U_A U_B$ as Case 1 and Case 3 of the function $maxxy$ defined in Definition 1. We know from Theorem 1 that, for the edge $U_B = -m_k U_A + \overline{U}_B^k$, the slope m_k is 1 and y-intercept $\overline{U}_B^k$ is $\overline{U}_B$. Substituting these values in the function $maxxy$ we get the above function. ■

C. Mapping of NBS Solution to a Bid Profile

There can be more than one bid profile s, that can yield the utility $\left(\frac{\overline{U}_B}{2}, \frac{\overline{U}_B}{2}\right)$.

We know from Theorem 3 that, there are two possibilities for NBS solution (U_A^*, U_B^*). It is trivial from Lemma 2 that, the bid profile $\overline{s}$ yields utility pair $(\overline{U}_A, U_B^{'})$ which is one of the possibility.

For the second possibility, we identify two bid profiles $\overline{s}$ and $\underline{s}$, which when played with probability p_1 and p_2 respectively, yields the utility pair $\left(\frac{\overline{U}_B}{2}, \frac{\overline{U}_B}{2}\right)$. The bid profile $\underline{s}$ can be identified by an optimization problem defined in the Definition 2 below.

Definition 2: Let us define the bid profile $\underline{s}$ as an outcome of the following optimization problem

$$\underline{s} = \underset{s \in S}{argmin}\, U_A(s)$$

subject to

$$U_A(s) + U_B(s) = \overline{U}_B \, \forall s \in S$$
$$U_A(s) = \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) p_i \right]$$
$$U_B(s) = \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{m} \beta_j y_{ij} \right) (\overline{s}_i - p_i) \right]$$
$$y_{ij} \in [0,1] \, \forall i \in N, \forall j \in M,$$
$$\sum_{i=1}^{n} y_{ij} \leq 1 \, \forall j \in M,$$
$$\sum_{j=1}^{m} y_{ij} \leq 1 \, \forall i \in N,$$
$$p_i \geq 0 \, \forall i \in N$$

The above definition says, select the bid profiles s, such that sum of the auctioneer utility (U_A) and the aggregated bidders' utility (U_B) is $\overline{U}_B$ and utility of auctioneer (U_A) is minimum among such bid profiles.

The two bid profiles, $\overline{s}_i$ and $(s_1, 1, 0, \cdots, 0)$ discussed in Theorem 1, satisfy the constraints of this optimization problem. Hence, this optimization problem has at least two feasible bid profiles satisfying the constraints.

Given the bid profiles $\underline{s}$ and $\overline{s}$, the probability p_1 and p_2 with which these profiles are to be played can be found by solving the equations below.

$$p_1 + p_2 = 1$$
$$p_1 U_A(\overline{s}) + p_2 U_A(\underline{s}) = \frac{\overline{U}_B}{2}$$
$$p_1 U_B(\overline{s}) + p_2 U_B(\underline{s}) = \frac{\overline{U}_B}{2}$$

Thus the bid profile $\underline{s}$ found by solving the above optimization problem in Definition 2 and the bid profile $\overline{s}$ when played with probability p_1 and p_2 respectively, yields the utility pair $\left(\frac{\overline{U}_B}{2}, \frac{\overline{U}_B}{2} \right)$.

D. An Illustrative Example

Let the input for the bid optimizer be $\overline{s}_1 = 1, \overline{s}_2 = 2$, $\overline{s}_3 = 4$. and $S_1 = \{0, 1\}$, $S_2 = \{0, 1, 2\}$ $S_3 = \{0, 1, 2, 3, 4\}$. Let $\beta_1 = 0.85$, $\beta_2 = 0.75$. Substitute $\overline{s}_1, \overline{s}_2$, and $\overline{s}_3$ to compute $\overline{U}_B$ and $\overline{U}_A$.

$$\overline{U}_B = \sum_{i=1}^{m} \beta_i \overline{s}_i = 4\beta_1 + 2\beta_2 = 4.9$$

$$\overline{U}_A = \sum_{i=1}^{m} \beta_i \overline{s}_{i+1} = 2\beta_1 + 1\beta_2 = 2.45.$$

In this case, $\overline{U}_A = \frac{\overline{U}_B}{2}$, $\overline{s} = (1, 2, 5)$. The NBS solution (U_A^*, U_B^*) (2.45,2.45). Hence, the the bid profiles $\overline{s}$ corresponds to the Nash Bargaining Solution of the auctioneer and the aggregated bidders.

IV. Conclusion and Future Work

Bid optimizers have recently become popular in the arena of sponsored search auctions, driven by the need to retain advertisers, in the face of increasing competition among search engine companies. In this paper, we have presented a novel bid optimization algorithm, based on a two person Nash bargaining model, where, the auctioneer is one player and a virtual aggregated agent representing all the n bidders is the other player. We have shown that the proposed bid optimization algorithm yields a little less revenue to the auctioneer, but leads to increased retention of advertisers.

This paper opens up many interesting challenges. We have taken the underlying sponsored search auction mechanism as GSP mechanism. We would like to observe the changes in the properties of the mechanism when the underlying auction mechanism is changed. We also want to relax the condition bidder independent click through rate and study its impact.[4]

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