Optimal Auctions for Multi-Unit Procurement

Raghav Kumar Gautam  
raghavg@csa.iisc.ernet.in  
Student, ME(ISE)  
E-Commerce Lab  
CSA Department, IISc

Prof. Y. Narahari  
hari@csa.iisc.ernet.in  
Research Supervisor  
E-Commerce Lab  
CSA Department, IISc

Abstract

Our attention is focused on designing procurement mechanisms which a buyer can use for procuring multiple units of a homogeneous item based on bids submitted by autonomous, rational, and intelligent suppliers. In this document, we focus on the case when the buyer wants to minimize his procurement costs. Optimal mechanisms accomplish this while keeping in view (1) Bayesian incentive compatibility (which ensures that truthful bidding is a best response for each supplier whenever all the suppliers are bidding truthfully) (2) Individual rationality (which ensures that the suppliers obtain non-negative payoffs by participating in the auction). We use the framework of games with incomplete information for analysis of our problem. We look at the case of volume discounts in detail and also provide several directions for future work.

Keywords: Mechanism Design, Optimal Auction, Incentive Compatibility, Individual Rationality, Direct Revelation Mechanism, Rational Supplier, Multiunit, Procurement Auction, Reverse Auction.

Acronyms: Following acronyms have been freely used throughout this report.  
BIC Bayesian Incentive Compatible  
BNE Bayesian Nash Equilibrium  
DSIC Dominant Strategy Incentive Compatible  
IC Incentive Compatibility  
IR Individual Rationality  
SCF Social Choice Function  
IPV Independent Private Values  
VCG Vickrey Clarke Groves  
dAGVA d’Aspremont Gerard Verat Kenneth Arrow  
NP Non-deterministic Polynomial

Introduction

Procurement auctions are market mechanisms where a set of goods is to be purchased. For the same reason, it is also known as reverse auctions. In their survey paper Chandrashekhar, Narahari, Rosa, Charles, Kulckarni, Dayama and Tew [1], have given a very detailed view of procurement auctions. Procurement auctions are widely used by the Government as well as different companies for purchasing goods. Bidders in any auction setting cannot be expected to truthfully reveal their cost of producing goods, unless doing so gives them maximum possible profit. Game theory being a set of models of conflict and cooperation for multiple entities, comes in very handy for analyzing and solving such problems.

1.1 Background

A buyer in a procurement auction, would like to minimize the total cost of procurement of the required goods. In order to accomplish this, he would like to choose his auction mechanism in such a manner that firstly, the suppliers bid truthfully and then he must solve the underlying optimization problem.

Assumptions about the suppliers Typical assumption that we can make about the suppliers is that they are autonomous, rational, and intelligent. By autonomous, we mean that the supplier is not under control of the buyer or any of the other supplier and hence, can take his own decision. For taking decisions rationality in the game theoretic sense is another important criterion, which means that the suppliers would always take decision according to their own objectives. Selfishness is an important implication of rationality which implies, that the supplier would always try to maximize his profit out of the auction. Intelligence means, that every supplier has enough computational resources to find what is in his best interest.

Notation Notation given in Table 1 has been used throughout this report. Additional notation wherever required has been explicitly explained.
Table 1: Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>Number of suppliers</td>
</tr>
<tr>
<td>$N$</td>
<td>Set of suppliers represented by $1, 2, ..., n$</td>
</tr>
<tr>
<td>$i$</td>
<td>Index of suppliers</td>
</tr>
<tr>
<td>$j$</td>
<td>Index of supplier's capacity intervals</td>
</tr>
<tr>
<td>$m$</td>
<td>Number of identical items which the buyer wishes to procure</td>
</tr>
<tr>
<td>$R$</td>
<td>Total revenue to the buyer by procuring $m$ items</td>
</tr>
<tr>
<td>$m_i$</td>
<td>Maximum number of units of items which the supplier $i$ can supply</td>
</tr>
<tr>
<td>$m_i^*$</td>
<td>Maximum number of units of items which the supplier $i$ has reported that it can supply</td>
</tr>
<tr>
<td>$I_i$</td>
<td>Set of price intervals for supplier $i$</td>
</tr>
<tr>
<td>$I_i^*$</td>
<td>Set of price intervals reported by the supplier $i$</td>
</tr>
<tr>
<td>$q_{ij}$</td>
<td>Capacity announced by supplier $i$ for $j$th interval</td>
</tr>
<tr>
<td>$q_i$</td>
<td>Total capacity of the supplier $i$ represented by $q_i = \sum_{j \in I_i} q_{ij}$</td>
</tr>
<tr>
<td>$c_{ij}$</td>
<td>True value of per unit cost for supplier $i$ for $j$th interval</td>
</tr>
<tr>
<td>$c_i$</td>
<td>Lower limit of the per unit cost for supplier $i$</td>
</tr>
<tr>
<td>$\bar{c}_i$</td>
<td>Upper limit of the per unit cost for supplier $i$</td>
</tr>
<tr>
<td>$b_i$</td>
<td>True type of supplier $i$ represented by the ${c_{ij}, q_{ij}}, \forall j \in I_i$</td>
</tr>
<tr>
<td>$b_{-i}$</td>
<td>True types of all the suppliers other than $i$ represented by $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$</td>
</tr>
<tr>
<td>$b_{-i}^*$</td>
<td>Type announced by the supplier $i$ represented by ${\hat{c}_i, \hat{q}_i}$</td>
</tr>
<tr>
<td>$B_i$</td>
<td>Set of possible types of supplier $i$</td>
</tr>
<tr>
<td>$B_{-i}$</td>
<td>Set of possible types of all the supplier other than $i$ represented by $B_1, B_{i-1}, B_{i+1}, \ldots, B_n$</td>
</tr>
<tr>
<td>$b$</td>
<td>A particular type profile</td>
</tr>
<tr>
<td>$b'$</td>
<td>A profile of bids of the suppliers</td>
</tr>
<tr>
<td>$B$</td>
<td>Set of possible types profiles represented by $B = \times_{i \in N} B_i$</td>
</tr>
<tr>
<td>$f(\cdot)$</td>
<td>Social choice function</td>
</tr>
<tr>
<td>$x(\cdot)$</td>
<td>Allocation rule of $f(\cdot)$</td>
</tr>
<tr>
<td>$X_i(\cdot)$</td>
<td>Expected allocation by $f(\cdot)$ to the supplier $i$</td>
</tr>
<tr>
<td>$l_i(\cdot)$</td>
<td>Payment to supplier $i$</td>
</tr>
<tr>
<td>$T_i(\cdot)$</td>
<td>Expected payment by $f(\cdot)$ to the supplier $i$</td>
</tr>
<tr>
<td>$U_i$</td>
<td>Expected utility of supplier $i$</td>
</tr>
<tr>
<td>$p_i(b_i)$</td>
<td>Surplus offered when $i$ announces $b_i$ as his type</td>
</tr>
</tbody>
</table>

**Incentive Compatibility** In any arbitrary auction, it cannot be guaranteed that the suppliers would bid truthfully because it might be better for them to provide false information. By incentive compatibility, we refer to the property that telling truth becomes a best response for all the suppliers. Incentive compatibility comes in two flavors: (1) Dominant Strategy Incentive Compatible (DSIC) (2) Bayesian Incentive Compatible (BIC). By DSIC mechanisms, we mean auctions in which every bidder finds revealing his true value to be a best strategy regardless of what other suppliers are bidding. An example of DSIC mechanism is the famous VCG mechanisms described by Clarke[2], Vickrey[3] and Groves[4]. In BIC mechanisms, every bidder would find revealing his true value to be the best strategy given that other suppliers are bidding truthfully. Examples of BIC include dAGVA mechanisms described by d’Aspremont and Gérard-Varet [5], Arrow [6] and Optimal Auctions described by Myerson [7]. Clearly DSIC is much stronger condition and costly to insure. BIC is not that demanding and can still ensure truthful bidding, and at the same time giving the seller a chance to reduce procurement cost further. In this report by incentive compatibility, we would always mean BIC unless stated otherwise.

**Individual Rationality** Since the suppliers cannot be forced to participate in the auction, they should have some incentive to participate in the auction that is, it should never happen that a supplier is worse off by participating in the auction. More simply, it can be said that the suppliers should always get something as profit by participating in the auction. Individual rationality comes in three versions (1) Ex-ante IR (2) Interim IR (3) Ex-post IR. Ex-ante IR refers to the individual rationality even before the suppliers find out their costs or the costs of other suppliers for the suppliers. Interim IR refers to the individual rationality after suppliers find out their true costs and know nothing about the costs of other players but have not yet placed their bids yet. Ex-post IR refers to the rationality after allocation has been done by the auction and the payments have been determined. Clearly, Ex-post IR is the strongest of the three conditions followed by Interim IR and then Ex-ante IR. In this report by IR we mean Interim IR.

**Optimal Mechanism** A mechanism which minimizes the cost of procurement while ensuring individual rationality and incentive compatibility is known as optimal mechanism. In the book by Mas-Colell, Whinston, Michael and Green [8] and in the technical report of Garg [9], a fair amount mechanism design
including optimal auctions has been discussed.

1.2 Motivation and Contributions

We are motivated to do this project because of its wide applicability. Optimal mechanisms are highly desirable because they extract the highest payoff possible out of the auction for the auctioneer. Myerson [7] in his paper in 1981, proposed an optimal auction for selling a single indivisible good, but still there are many practical situations for which optimal mechanisms are yet to be designed. We look into some interesting, compelling situations in procurement auctions and attempt to design optimal auctions in those situations.

Recently, several authors have extended optimal auctions to interesting practical situations. Examples include: Hastagiri and Narahari [10], Malakhov and Vohra [11], Iyengar and Kumar [12]. The specific contributions of this project would be: (1) Designing an optimal auction in the case of procurement of multiple units of a single item with volume discounts (2) Design of optimal auctions in the case of procurement of items with multi-dimensional bids (3) Design of optimal auctions for the case of combinatorial procurement auctions.

1.3 Outline

The outline of the rest of this report is as follows. In Section 2, we would be discussing a review of relevant work already done in this area, summarizing related work and the research gap present. In Section 3, we present a line of attack for the problem that we have undertaken. In Section 4, we present a summary and our future line of work.

2 Review of Relevant Work

The area of optimal auction took off with the historic paper [7] of Myerson in 1981. Myerson designed an optimal auction for single indivisible item that has to be sold. A single seller and multiple buyers are present in the system. Aim of the seller is to extract as much revenue from the auction as possible. Myerson in his work also considered uncertainties that might be playing a role in the bidding. These uncertainty might be of two types (1) preference uncertainty (2) quality uncertainty. Preference uncertainty is when the buyers have different valuations for the same object and quality uncertainty is when the bidders have different opinion about the quality of the item itself. This uncertainty leads to revision effect that is, the bidders might want to change their valuation of the item based on the valuation of others. Myerson also gave a characterization of feasible auction mechanisms based on IR an IC and the fact that only one item is there to be auctioned. Myerson also came up with virtual valuation function. He then defined regularity condition for optimal auction as the case, when virtual valuations are monotonically strictly increasing function of \( t_i \). He came up with allocation scheme and payment scheme. He proved that bidders must be made to pay only when he gets the object and must be made to pay the smallest bid amount that would have won the auction for him. Myerson also showed that optimal mechanism need not be social welfare maximizing by means of an example.

In [11], Malakhov and Vohra have explored connection between continuous and discrete approach to optimal auction with Multi dimensional types. They have taken the types in discrete space and map the problem of optimal auction to a network flow problem in dual interpretation by introducing one node for each type and cleverly assigning the costs to the edges. This gives us a network in which a negative cycle implies dual to be unbounded and hence the primal becomes infeasible. Shortest path to a vertex corresponding to a particular type gives the expected payment for that type.

In [10], Hastagiri and Narahari have discussed different incentive compatible mechanisms for multi-unit procurement scenario. They have considered two-dimensional bid structure having just a cost per unit of item and the maximum quantity that can be supplied by the supplier. They have proposed (1) P-DSIC (2) P-BIC (3) P-OPT. P-DSIC guarantees that truthful bidding becomes best response for each supplier in dominant strategies. This mechanism belong to the VCG class of mechanisms and hence satisfies IC, it is also IR. It is robust and bidding logic for each player is trivial, but it result in high procurement cost. It also requires absence of critical supplier and is not budget balanced. Then they proposed P-BIC as a Bayesian Nash Incentive Compatible mechanism. This mechanism belongs to dAGVA class of mechanisms. It overcomes both the limitations of the P-DSIC. But, P-BIC mechanism does not ensure individual rationality. Finally, they have proposed P-OPT mechanism, which is both IR as well as IC and tries to minimized the procurement cost for the buyer. They have approached the problem on the lines of Myerson [7]. After first developing characterization for a feasible mechanism for
their problem, they have defined virtual cost function. They have made assumption similar to Myerson’s regularity assumption and finally they have come up with an allocation and corresponding payment scheme for the optimal auction. They have rigorously proved the validity of the P-OPT.

2.1 Research Gap

After scanning a plethora of literature, it was found that:

- Kameshwaran, Narahari, Rosa, Kulkarni, Tew in [13], and Kameshwaran in [14], have done a lot of work in the area with regard to computational issues of winner determination problems. Their work is relevant when the induced game by the mechanism is a game with complete information. But they have not taken into account issues of incentive compatibility and individual rationality, that makes their work inefficacious for the setting when the induced game is a Bayesian game.

- Hastagiri and Narahari in [10], have solved the optimal mechanism design problem for restricted special cases of procurement auctions. They have considered the auctions for procuring multiple units of a single item. The bid structure they allow has a useful but rigid structure of allowing only a single cost per unit for all the units. In practice, the suppliers tend to charge less per unit when bulk purchase is made this is called volume discount, and it is a common practice. Optimal auctions for more realistic settings of volume discounts have to be designed.

- Myerson in [7], has designed an optimal auction for the setting of selling of a single indivisible item to a number of rational buyers. A lot of extension and application of Myerson’s work for different settings are required for procurement scenarios in the real world.

- In the most general procurement setting, we find that there are a number of different goods to be purchased, and the suppliers have different costs for different bundles of goods. Such a scenario is called combinatorial procurement. To the best of our knowledge, no work for designing optimal auction has been done in the area of combinatorial procurement.

- Another possible generalization is the case where items have different attributes and the buyer may want to pay more or less according to the attributes of the items. Optimal auction for procurement in multi attribute settings is yet to be designed.

3 Optimal Multi-unit Procurement Auctions

3.1 Setup

There is a single buyer in the system who wants to purchase \( m \) homogeneous items form \( n \) suppliers, the set of suppliers being \( N = \{1, 2, \ldots, n\} \). See Figure 1. Let \( R \) represent the total money required for making the procurement by the auction. In real life scenario, generally all the items need not be purchased from a single supplier and at the same time a single supplier may not be able to fulfill the demand of the buyer. The buyer would like to make his purchase in such a way, that the total amount that he has to pay for the items is minimized.

![Figure 1: Procurement scenario for a single buyer](image)

Let the maximum number of items that supplier \( i \) can supply be given by \( q_i \). But all these items may not be supplied at the same price, because of volume discounts what happens is that, the price per unit for the items for the supply decreases as the number if items ordered from a single supplier increases. See Figure 2. We assume these price per unit to be piece-wise constant, which is common in practice. The piece wise constant implies the whole pricing can be broken up into intervals for each of the supplier. Let \( I_i \) denote the set of price intervals for the supplier \( i \) and \( q_{ij} \)
be the capacity for the $i^{th}$ supplier for the $j^{th}$, $j \in I_i$ cost interval; cost per unit item for this interval be denoted by $c_{ij}$. Let $b_i \in B_j$ denote the true type that is $b_i = \{c_{ij}, q_{ij} ; j \in I_i\}$, note that this is a set. But at the time of submitting bid the supplier might want to submit $\hat{b}_i = \{\hat{c}_{ij}, \hat{q}_{ij} ; j \in I_i\}$ reporting the maximum items that he can supply to be $\hat{m}_i$ instead of $m_i$. Let $b_{-i} \in B_{-i}$ denote the true type and $b_{-i} \in B_{-i}$ denote the reported type of other players. The overall profile of bids is represented by $\hat{b} \in B$ which goes as input to the auction. We have assumed independent private value model, that is type distribution between different suppliers is independent and a supplier known only his type.

### 3.2 Multi - Unit Procurement with Complete Information

As we try to minimize the procurement cost even under the assumption that suppliers true values are known to the buyer. In [13], Kameshwaran, Narahari, Rosa, Kulkarni and Tew have taken a goal programming based approach to the problem. They have also shown that single item, single attribute, Multi-unit procurement with volume discount bids leads to piecewise linear knapsack problem. They have given exact as well as heuristics based methods for solving such problems. Kameshwaran in his PhD thesis [14] has also given a fully polynomial time approximation algorithm for piecewise linear knapsack problem. Under the setting of single-item, single attribute, multi-unit procurement auctions where the bidders use marginally decreasing, piecewise constant functions to bid for goods. In [15], Kothari, Parke and Suri have shown that winner determination problem is NP hard, by showing that it is generalization of classical knapsack problem. They also have given a fully polynomial time approximation algorithm for solving the problem and shown that their scheme is approximately strategyproof.

### 3.3 Multi - Unit Procurement with Incomplete Information

For the case where suppliers are capacitated Hastagiri and Narahari in [10] and Iyengar and Kumar in [12] have given examples that show that classical $K^{th}$ price auction does not ensure truthfulness. The $K^{th}$ price auction induces a weakly dominant strategy equilibrium for truthful revelation of cost values. But it does not stop them from falsifying their capacity component of the bid. By underbidding their capacity values the suppliers can create fictitious shortage of resources in the system forcing the buyer to pay extra for virtually limited resources. Design of optimal mechanism for multi-unit procurement is challenging. We start the discussion with the classical work of Myerson [7].

**Optimal Auction for Selling Single Indivisible Item**

Myerson in [7] designed optimal auction for selling a single indivisible item. The setup for the Myerson's work is as follows. Let the set of bidders be denoted by $N = \{1, \ldots, n\}$ and value estimate of Bidder $i$ by $t_i$. It has been assumed that probability density function for bidder $i$’s estimate of $t_i$ is continuous $f_i(t_i) > 0, \forall t_i \in [\underline{\theta}_i, \overline{\theta}_i], -\infty < \underline{\theta}_i < \overline{\theta}_i < \infty$ and the cumulative distribution function for $f(\cdot)$ is represented by $F_i(t_i) = \int_{-\infty}^{t_i} f_i(s_i) ds_i$. Set of all possible combinations of bidder’s value estimates is represented by $T = [\underline{\theta}_1, \overline{\theta}_1] \times \ldots \times [\underline{\theta}_n, \overline{\theta}_n]$. Set of all possible combination of value estimates of all players other that $i$ is $T_{-i} = [\underline{\theta}_{-i}, \overline{\theta}_{-i}]$ with $\forall j \in N, j \neq i$. Value estimates of the $n$ bidders being independent, their joint density function on $T$ for the vector $t = (t_1, \ldots, t_n)$ would be $f(t) = \Pi_{j \in N} f_j(t_j)$. Every player $i$ considers his own value estimate to be a known quantity, but will have to assess the the probability distribution for other bidders value estimate. Bidder $i$ would assess the joint density function on $T_{-i}$ for the vector $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ to be $f_{-i}(t_{-i}) = \Pi_{j \in N, j \neq i} f_j(t_j)$. Seller’s reservation price for the item is $t_0$ and is common knowledge but the bidder’s value estimates is not a common knowledge. One bidder’s value estimates may be unknown to seller and the other bidders because of the following:
1. Preference uncertainty: $i$ likes Monalisa $\Rightarrow i$'s value is not going to change because of valuation of other bidders.

2. Quality uncertainty: bidder's values may change as they learn about other bidders valuations.

Bidder $i$ on learning bidder $j$'s value to be $t_j$ would revise his own valuation by $e_j(t_j)$, this is called revised effect function. Therefore after $i$ learns $t = (t_1, \ldots, t_n)$ he would revise his own valuation of the object as $v_i = t_i + \sum_{j \in N, j \neq i} e_j(t_j)$ and the seller would revise his valuation to $v_0 = t_0 + \sum_{j \in N} e_j(t_j)$.

If $(p, x)$ be the mechanism in place then, The expected utility of $i$ from the auction mechanism $(p, x)$ is

$$U_i(p, x, t_i) = \int_{t_i}^{T} (v_0(t) - x_i(t)) f_{-i}(t-i) dt - v_i(t_i)$$

where $dt = dt_1 \ldots dt_{i-1} dt_{i+1} \ldots dt_n$

And the expected utility of seller from this auction mechanism is

$$U_0(p, x) = \int_T (v_0(t) - x_i(t)) f(t) dt$$

where $dt = dt_1 \ldots dt_n$

**Constraints on mechanism $(p, x)$** The mechanism to be feasible must satisfy

1. Since, there is only one object to be allocated $p$ must satisfy $\sum_{j \in N} p_j(t) \leq 1$ and $p_i(t) \geq 0$

2. Individual rationality implies $U_i(p, x, t_i) \geq 0, \forall i \in N, \forall t_i \in \theta_i$

3. Incentive compatibility implies $U_i(p, x, t_i) \geq U_i(p, x, s_i)$ where $U_i(p, x, s_i) = \int_{t_i}^{T} (v_0(t) - x_i(t)) f_{-i}(t-i) dt - v_i(t_i)$, $\forall i \in N, \forall t_i \in \theta_i$, $\forall s_i \in \theta_i$.

**Revelation Principle** For every feasible auction there always exists a Direct Revelation Mechanism. In Direct Revelation Mechanism seller first asks each bidder to announce his type, and then computes strategies of bidders and finally implements the outcomes prescribed in the given auction game for these strategies. The DRM can be described by a pair of outcome functions

$$(p, x): p : T \rightarrow R^n, x : T \rightarrow R^n$$

such that if $t$ is the announced value estimate of all the bidders then,

- $p_i(t)$ is the probability that that $i$ gets the object and
- $x_i(t)$ is the expected amount of money which $i$ must pay (even if he did not get any object)

**Seller’s Problem** Choose functions $p : T \rightarrow R^n$ and $x : T \rightarrow R^n$ so as to maximize $U_0(p, x)$ subject to above constraints on the mechanism. Let $Q_i(p, t_i)$ be the conditional probability that bidder $i$ will get the object from the auction mechanism $(p, x)$ given his value estimate is $t_i$ thus, $Q_i(p, t_i) = \int_{T \rightarrow} p_i(t) f_{-i}(t-i) dt - i$.

**Claim:** $(p, x)$ is feasible iff

1. $s_i \leq t_i$ then $Q_i(p, s_i) = Q_i(p, t_i), \forall i \in N, \forall s_i, t_i \in \theta_i$

2. $U_i(p, x, t_i) = U_i(p, x, \theta_i) + \int_{s_i}^{t_i} Q_i(p, s_i) ds_i, \forall i \in N, \forall t_i \in \theta_i$

3. $U_i(p, x, \theta_i) \geq 0, \forall i \in N$

4. $\sum_{j \in N} p_j(t) \leq 1$ and $p_i(t) \geq 0, \forall i \in N, \forall t \in T$

**Allocation and payment for the Optimal Auction** Suppose $p : T \rightarrow R^n$ maximizes

$$\int_T (\sum_{i \in N} (t_i - c_i(t_i) - \frac{1-E_i(t_i)}{f_i(t_i)} - t_o)) p_i(t) f(t) dt$$

Subjected to constraints 2 and 4 mentioned previously. Also suppose that,

$$x_i(t) = p_i(t) v_i(t) - \frac{1}{f_i(t)} \int_T p_i(t-i, s_i) ds_i, \forall i \in N, \forall t \in T$$

then $(p, x)$ represents an optimal auction. And the total revenue obtained by the seller from this auction is

$$U_0(p, x) = \int_T (\sum_{i \in N} (t_i - t_0 - c_i(t_i) - \frac{1-E_i(t_i)}{f_i(t_i)}) p_i(t) f(t) dt + \int_T v_0(t) f(t) dt - \sum_{i \in N} U_i(p, x, \theta_i))$$

The above equation also implies that seller's expected utility from an auction mechanism depends totally on probability function $p$ and $U_i(p, x, \theta_i)$.

**Optimal auction in Regular case** Problem is regular, if $c_i = t_i - c_i(t_i) - \frac{1-E_i(t_i)}{f_i(t_i)}$ is a monotone strictly increasing function of $t_i$ for all $i \in N$. i.e. the problem is regular if $c_i(s_i) < c_i(t_i)$ whenever $\theta_i \leq s_i < t_i \leq \theta_i$. Now, consider the auction mechanism where the seller keeps the object if $t_0 > max(c_i(t_i))$ and gives it to the bidder with highest $c_i(t_i)$ otherwise. In this scheme $p_i(t) > 0 \Rightarrow c_i(t_i) = max_{j \in N} (c_j(t_j)) \geq t_0$. For all $t \in T$, this mechanism maximizes the sum
\[ \sum_{i \in N} (c_i(t_i) - t_i) p_i(t) \]

subjected to \[ \sum_{i \in N} p_j(t) \leq 1 \] and \[ p_i(t) \geq 0, \forall i \]

Thus \( p \) maximizes
\[ \int_T \left( \sum_{i \in N} (t_i - c_i(t_i) - \frac{1-F(t_i)}{f(t_i)} - t_0) p_i(t) f(t) dt \right) \]

subjected to \[ \sum_{j \in N} p_j(t) \leq 1 \] and \[ p_i(t) \geq 0, \forall i \in N, \forall t \in T \]

For feasibility, we need to check if \( s_i \leq t_i \implies Q_i(p, s_i) \leq Q_i(p, t_i), \forall i \in N, \forall s_i, t_i \in [\hat{p}_i, \hat{q}_i] \)

Observe that \( c_i(s_i) < c_i(t_i) \), so whenever bidder \( i \) could win the object by submitting \( s_i \) as his value estimate then he could also win by submitting \( t_i \). That is \( p_i(t_{-i}, s_i) \leq p_i(t_{-i}, t_i) \, \forall t_{-i} \). So \( Q_i(p, t_i) \) the probability of winning the object is increasing function of \( t_i \). Thus, \( p \) is feasible.

Allocation and payment scheme are defined in the following fashion gives us an optimal auction.

\[
p_i(t) = \begin{cases} 
1 & \text{if } t_i > z_i(t_{-i}) \\
0 & \text{if } t_i < z_i(t_{-i}) 
\end{cases}
\]

\[
x_i(t) = \begin{cases} 
z_i(t_{-i} + \sum_{j \in N, j \neq i} e_j(t_j)) & \text{if } p_i = 1 \\
0 & \text{if } p_i = 0 
\end{cases}
\]

where,
\[ z_i(t_{-i}) = \inf \{ s_i | c_i(s_i) \geq t_0 \land c_i(s_i) \geq c_j(s_j), \forall j \neq i \} \]

That is bidder \( i \) pays only when he gets the object, and then he pays \( v_i(t_{-i}, z_i(t_{-i})) \) which is the lowest amount that would have won the bid for him.

**Optimal Auction for Procuring Multiple Units of a Single Item**

Hastagiri and Narahari in [10] have extended Myerson’s work for the case of multiple units of a single item in procurement scenario. The setup for this is same as that we have discussed at the beginning of this section, the only difference being that since they have not considered the case of volume discount the price interval set of each supplier is singleton and hence subscript \( j \) with cost and capacity variables would be dropped.

**Characterization of Optimal Auction** In [10], Hastagiri and Narahari have developed characterization for their optimal auction as: an allocation rule \( x \) is BIC and IR if the expected allocation \( X_i(c_i, q_i) \) is non-increasing in cost valuation \( c_i \) and the offered surplus \( p_i(\hat{c}_i, \hat{q}_i) \) is of the form

\[
p_i(\hat{c}_i, \hat{q}_i) = p_i(\hat{c}_i, \hat{q}_i) + \int_{\hat{c}_i}^{\pi} X_i(y, \hat{q}_i) dy
\]

And \( p_i(\hat{c}_i, \hat{q}_i) \) must be non-negative and non-decreasing in \( \hat{q}_i \).

**Virtual Cost Function** They have defined virtual cost function as
\[
H_i(c_i, q_i) = c_i + \frac{F_i(c_i | q_i)}{f_i(c_i | q_i)}
\]

They use these \( H_i(c_i, q_i) \) values to compute the assignment. They order the suppliers on the basis of their \( H_i(c_i, q_i) \) values and keep allocating units to them at their full capacity in decreasing order of \( H_i \) values until all units are allocated. Let \( \hat{p} \) denote the index of the supplier that satisfies,
\[
\sum_{j=1}^{[\hat{p}]-1} q[j] < m
\]
\[
\sum_{j=1}^{[\hat{p}]} q[j] \geq m
\]

**Allocation and Payment** Then, the allocation function is given as
\[
x[j] = \begin{cases} 
\hat{q}[j], & [i] < [\hat{p}] \\
m - \sum_{j=1}^{[\hat{p}]-1} \hat{q}[j], & [i] = [\hat{p}] \\
0, & \text{otherwise}
\end{cases}
\]

And the payment function is given as
\[
t_i(\hat{b}) = c_i x_i(\hat{b}) + \int_{c_i}^{\pi} X_i(y, \hat{q}_i) dy
\]

**4 Optimal Procurement Auction with Volume Discount Bids**

The setup for the remaining discussion remains the same as that of the previous section. We have some preliminary results for a bottom up attack on the problem that we will discuss in this section. Under the assumption that we have a characterization and virtual cost function in place for our setting, we will give an important result with regard to the allocation.

**Proposition 1:** There always exists an allocation in the optimal auction such that all the suppliers with the possible exception of one, would be supplying to their maximum capacity.

**Proof for Proposition 1:** We will show that for any allocation that is optimal, we can find an allocation
which satisfies the above property or the allocation itself is not optimal. Let us say we have an optimal allocation, where there are \( k \) suppliers who are not supplying to the maximum of their capacity. Without loss of generality, we can assume that they are 1, 2, \ldots, \( k \) and let their cost per unit for the last interval be \( c_{1j_1}, c_{2j_2}, \ldots, c_{kj_k} \) respectively.

**Proposition 1.1:** \( c_{1j_1} = c_{2j_2} = \ldots = c_{kj_k} \).

**Proof:** For contradiction, let us assume that there are some suppliers for which the above claim is not satisfied. Now we can find \( x, y \in 1, 2, \ldots, k \) such that \( c_{xj_x} > c_{yj_y} \). Since \( y \) is not allocated to its full capacity, we can unallocate a unit from \( x \) and allocate it to \( y \) to get a less procurement cost hence, the allocation is not optimal giving us the required contradiction.

To continue the proof of Proposition 1, we will use the same trick again. This time we unassign units from 1 and assign it to 2. This can never give us a greater procurement cost as the bids follow volume discount. We keep repeating this till 2 is allocated to his full capacity. We do similar reallocation from 1 to 3, \ldots, \( k \). The allocation thus obtained leaves only one supplier namely 1 partially allocated and the rest are fully allocated. This proof is only existential and necessary to show the validity of the allocation method explained next.

**Allocation** Let \( H_i(b_i) \) represent the virtual cost function. Then allocation can be done by ordering suppliers on the basis of their \( H_i(b_i) \) values, and then allocate them units to their full capacity in the order of decreasing \( H_i \) values. The allocation for the optimal auction can be done in this fashion. Having given the allocation scheme, all that needs to be done to solve this problem is to come up with the *virtual function*.

5 **Future Work**

In our future work:

- We shall complete the designing of the optimal auction for multi-unit procurement with volume discounts.
- We would then attempt to extend the work for the setting where purchase decisions for the item also need to consider the different attributes of the items.
- We would endeavor to design optimal combinational procurement auctions.

**References**


