Stability and Efficiency of Social Networks with Strategic, Resource Constrained Nodes

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***Abstract*—Recently, the topic of social network formation has received significant attention since the structure of the networks has a profound impact on the economic outcomes in many real world applications such as large exchange markets, sponsored search auctions, and viral marketing. Stability and efficiency are two important properties which are sought in such networks. These two properties are both desirable but not always compatible. This paper investigates the tradeoff between stability and efficiency in a noncooperative game model of social network formation. In our model, we consider network formation in which each node can form at most k links due to scarcity of the resources. We formulate the network formation process as a strategic form game. We view the notion of stability as obtaining a Nash equilibrium outcome and efficiency as maximizing the value of the network. In this setting, we show that all efficient networks are stable in both the cases: (i) $k = 1$ and (ii) $k = 2$.**

***Keywords*-Social networks, network formation, stability, efficiency, Nash equilibrium.**

I. Introduction

A social network is a social structure made of individuals or organizations that are tied by one or more specific types of interdependencies. We represent a social network with a graph model where typically each individual is represented by a node, and there is an edge between two nodes if there exists a social interaction between them. As the structure of the network can have a profound impact on the economic outcomes and also on the welfare of the participating entities, there has been a significant interest recently from research communities to understand the complex structures of the networks. Towards this end, we need to understand the process of link formation in social networks. We use the term *network formation* to mean that nodes in the network establish links to the other nodes to potentially gain by communication and information exchange.

We note that the nodes in the social network are individual entities. We assume that such individual entities are intelligent and rational in the sense that they maintain only those social relationships (or links to other nodes) with which their benefits (or utilities) are maximized. For this reason, nodes act strategically while choosing the set of nodes to which they form links. Hence we call the network formation process as *strategic network formation*.

Stability and efficiency are two important properties we seek in the network formation process. Informally, we call a network stable if it is in a strategic equilibrium and we call a network efficient if the sum of the benefits (or utilities) of the nodes is maximal. It is important to note that the set of stable networks are not always compatible with the set of efficient networks [9]. The intuitive reason for the above observation is that globally efficient network structures need not be efficient from an individual node perspective. Hence the main focus in the strategic network formation process, from the literature stand point, is to understand and study the set of networks which are efficient and those which are stable [9].

In this paper, our focus is to study the tradeoff between stability and efficiency in a strategic network formation model of social networks. The following is a brief discussion of the model.

In many real world social networks, nodes are severely constrained by limited availability of resources. So the nodes cannot afford to maintain too many links to the other nodes in the network. We model the scarcity of resources to nodes by specifying that each node can form at most k links, where k is a given parameter. In this setting, we propose to model the network formation process as a strategic form game. In such a game, the utility of each node is defined to depend on the benefits through the paths to other nodes and the costs to maintain the links. We consider Nash equilibrium [11] as the notion of stability and the sum of utilities of the nodes as the notion of efficiency. In this setting, we study the conditions under which a network is stable and efficient. We also refer to this model as the strategic form game model for social network formation.

A. Organization of the Paper

We organize the paper in the following way. We review the relevant work in Section II and summarize our contributions

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in this paper in Section II-A. We present a brief discussion on a few important network structures in Section III-A and then present our proposed model in Section III-B. We carry out our analysis for $k = 1$ in Section III-C and the same for $k = 2$ in Section III-D. We conclude the paper in Section IV with pointers to future work.

II. Relevant Work

The topic of strategic network formation is very rich and there exist many different models to study this phenomenon. The following are most relevant for our work. Myerson [11] formulates the network formation process as a non-cooperative game and characterizes the stable networks when the notion of stability is Nash equilibrium.

Jackson and Wolinsky [9] propose two paradigmatic models of network formation, namely connections model and co-author model. In their models, they consider that the formation of a link requires the consent of both the nodes involved, but severance can be done unilaterally. The authors propose the notion of pairwise stability to capture equilibrium networks. A network is said to be pairwise stable [9] if (a) no individual node has an incentive to sever a link and (b) no pair of nodes have an incentive to form a new link. They derive a characterization of stable and efficient networks and show in general that stable networks are not necessarily efficient. Finally, the authors design allocation rules — for example an equal split rule — with which the existence of a stable and efficient network is possible.

Bala and Goyal [2] propose a noncooperative model of network formation by considering that links generate externalities whose value depends on the level of decay associated with indirect links. One distinctive aspect of the model is that the costs of link formation are incurred only by the node which initiates the link. Using this model, the authors characterize the structure of equilibrium networks and also study the dynamics of network formation. They also show that under some conditions the equilibrium networks are also efficient.

Bloch and Jackson [3] propose a new model of network formation while allowing transfers among the nodes. Jackson [7] focuses on the design of allocation rules for the network formation process.

Motivated by the applications in social networks and peer-to-peer networks, Laoutaris et.al. [10] propose a bounded budget connection game in which each node has a fixed budget to buy the links. Here the objective of each node is to buy the links while satisfying budget constraints in order to minimize its sum of weighted distances to the remaining nodes. The authors show that determining the existence of a pure Nash equilibrium is NP-hard. Moreover, in a specific setting in which all nodes have similar costs and budgets, the authors show the existence of a pure Nash equilibrium of the game. However, the authors do not focus on studying the equilibrium networks that are efficient.

A comprehensive survey of the various models of social network formation are described in Jackson [8] and Goyal [6].

In the literature, there is a research gap in respect of studying stability and efficiency tradeoffs in social networks with resource constrained nodes. We address this gap in this paper.

A. Our Contributions

We summarize the contributions of the paper in the following. We present a strategic form game model of network formation, in which each node can form at most k links where k is a given parameter. Here we interpret stability as obtaining a Nash equilibrium outcome. In this setting, we perform analysis to understand the relationship between stable and efficient networks in the following two scenarios: (i) $k = 1$ and $k = 2$. In both the scenarios, we show that the efficient networks are stable.

Our work is different from the relevant work in the literature in the following way. The most relevant work in the literature is Jackson and Wolinsky [9], Chowdhury [4], and Laoutaris et.al. [10]. The analysis in Jackson and Wolinsky [9] assumes that a node can form any number of links and considers pairwise stability as the notion of equilibrium. In our case, we use Nash equilibrium as the stability notion. Chowdhury [4] models the network formation process as a sequential-move game and naturally uses subgame-perfect Nash equilibrium as the notion of stability and analyzes the network structures where each node can form only one link. In our model, we work with a strategic form game and allow a node can form at most k links and we do a detailed analysis for the cases $k = 1$ and $k = 2$. The focus of Laoutaris et.al. [10] is to determine the existence of pure strategy Nash equilibrium and does not address the relationship between efficiency and stability of the networks.

III. A Strategic Form Game Model of Social Network Formation

A. A Few Important Network Topologies

Here we briefly explain a few example network structures that are useful in later analysis. Consider the network topologies shown in Figure 1. The network in Figure 1(i) is called a star network. The middle node in a star network

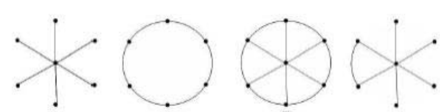

Figure 1. Network structures: (i) star network, (ii) cycle network, (iii) wheel network, and (iv) a wheel of length 3 with a local star

is called the center node and the rest of the nodes are called peripheral nodes. The network in Figure 1(ii) is called a cycle network and we call the network in Figure 1(iii) a wheel network. The node that is connected to the rest of the nodes in the wheel network is called the center and the rest of the nodes are called the peripheral nodes. We call the network in Figure 1(iv) a wheel of length 3 with a local star (WS_3) following the notation in Chowdhury [4].

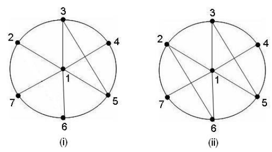

Figure 2. Network structures: (i) Extended wheel network with one link (EWN1), (ii) Extended wheel network with two links (EWN2)

Now consider another set of example network topologies with 7 nodes shown in Figure 2. We call the network in Figure 2(i) an extended wheel network with one link (EWN1). Such a network is a wheel network with one link added between a pair of non-adjacent nodes. The network in Figure 2(ii) is called an extended wheel network with two links (EWN2). Such a network is a wheel network with two links added between two pairs of non-adjacent nodes.

B. The Model

Let $N = \{1, 2, \ldots, n\}$ be the set of nodes that are involved in the process of social network formation. We model the link formation process as a strategic form game in the following way. Let k be a parameter that specifies the maximum number of links a node can form. A strategy s_i of node i is a subset of nodes with which it is willing to establish links and the cardinality of the subset is at most k. Note that it is possible node i may not establish any link, that is, s_i can be an empty set. Since a node cannot form a link to itself, $i \notin s_i$. Let S_i be the set of all possible strategies of node i, then it is clear that the cardinality of S_i is:

$$|S_i| = \sum_{j=1}^{k} \binom{n-1}{j} + 1$$

Let $s = (s_1, s_2, \ldots, s_n)$ be a strategy profile where s_i is the strategy of node i. Note that the strategy profile s can also be represented as $s = (s_i, s_{-i})$ where s_{-i} represents the profile of strategies of the nodes except node i. Assume that S is the set of all possible strategy profiles. That is, $S = S_1 \times S_2 \times \ldots \times S_n$.

When node i establishes a link to node j, we assume that it does not require the consent from node j and it incurs a cost c_i to node i to maintain the link. Once node i establishes a link to node j, then the link can also be used by node j; that is, we assume the links are undirected. Let $g(s)$ be the undirected graph that emerges due to the strategy profile s. $\forall i, j \in N$, assume that d_{ij} represents the length of the shortest path from node i to node j and if there is no path between two nodes i and j, then d_{ij} is defined to be infinity. We define the utility function $u_i : S \to \mathbb{R}$ of node i as follows:

$$u_i(s_1, s_2, \ldots, s_n) = \sum_{j \in N \setminus \{i\}} \delta_i^{d_{ij}} - |s_i| c_i$$

where δ_i is the utility that node i derives from another node through a direct link. Each $\forall i \in N$, δ_i is normalized to lie in the range $0 < \delta_i < 1$ so that $\delta_i^{d_{ij}}$ represents the utility that i derives by being connected to j through a shortest path of length d_{ij}. This captures the fact that the utility that nodes derive from near nodes is greater than from distant nodes. The utility of node i is the sum of benefits it gets from the rest of the nodes subtracted from its link maintenance costs. Note that this model is similar to that of Jackson and Wolinsky [9]. To make the analysis simple, we assume throughout this model that $\delta_i = \delta, \quad \forall i \in N$ and $c_i = c, \quad \forall i \in N$. The above framework clearly defines a strategic form game for the link formation process represented by

$$\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle.$$

Let Ψ be the set of all possible undirected graphs on N that emerge due to the strategy profiles in S. That is,

$$\Psi = \{g(s) : s \in S\}.$$

Now we recall a value function $v : \Psi \to \mathbb{R}$ that specifies a value for each graph $g(s) \in \Psi$:

$$v(g(s)) = \sum_{i \in N} u_i(s).$$

The above definition is simple and natural.

We now define two notions namely *efficiency* and *stability* based on $v(.)$. An undirected graph $g(s) \in \Psi$ is efficient if $v(g(s)) \geq v(g(s'))$ for all $s' \in S$. An undirected graph $g(s) \in \Psi$ is said to be stable if the corresponding strategy profile $s = (s_i, s_{-i})$ is in pure Nash equilibrium; that is,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \ \ \forall s_i' \in S_i, \ \forall i \in N.$$

If there is no confusion, we use the term *graph* to refer to *undirected graph*. It is clear from the literature and from intuition that there is a tradeoff between stability and efficiency. We now proceed to study this tradeoff. Throughout our analysis, we assume that $c < \delta$.

Lemma 1: When k is not bounded and $\delta - c > \delta^2$, then the unique efficient and stable network is the complete graph.

Proof: Please refer to Jackson and Wolinsky [9] for this proof. ∎

Now we analyze the network formation process when $k = 1$ and $k = 2$.

C. Scenario 1: $k = 1$

In this section, we consider the case that each node can form at most one link. In this setting, we first characterize the structure of the efficient networks.

Lemma 2: When each node can form at most one link and $c < \delta$, the efficient network following the proposed strategic form game model is

1) a star consisting of all nodes if
$$2(\delta - \delta^2) < c < 2\delta + (n-2)\delta^2$$
2) a wheel of length 3 with a local star (WS_3) when $c < 2(\delta - \delta^2)$.

Proof: It is given that $c < \delta$. Note that there must be at least $(n-1)$ edges in any graph otherwise there exists one isolated node which can improve its utility by forming a link. For this reason, we examine efficient network structure among the networks with $(n-1)$ links and n links respectively. Towards this end, we consider the following two claims.

Claim 1: A star is the efficient network structure among the networks with $(n-1)$ links.
The arguments from Jackson and Wolinsky [9] help to prove this. If the value of the star network be represented by $v(STAR)$; then,

$$v(STAR) = 2(n-1)\delta - (n-1)c + (n-1)(n-2)\delta^2$$

The value of such a star network must be non-negative. That is, $2(n-1)\delta - (n-1)c + (n-1)(n-2)\delta^2 > 0$ and this implies that $\Rightarrow \quad c < 2\delta + (n-2)\delta^2$.

Claim 2: A wheel of length 3 with a local star (WS_3) is the efficient network structure among the networks with n links.
The proof of this claim can be derived with the help of the arguments from Chowdhury [4].

If the value of the WS_3 be represented by $v(WS_3)$; then,

$$v(WS_3) = 2n\delta - nc + 2(n-3)\delta^2 + (n-3)(n-2)\delta^2$$

Now if $v(STAR) > v(WS_3)$, then $c > 2(\delta - \delta^2)$. On similar lines, we can show that WS_3 is an efficient network when $c < 2(\delta - \delta^2)$. This completes the proof of the lemma. ∎

Now we show that the above two efficient structures are stable.

Lemma 3: Both the efficient networks, namely the star network and a wheel of length 3 with local star (WS_3), are also stable.

Proof: We exhibit at least one strategy profile of the nodes that forms an efficient network structure and it is also in Nash equilibrium. We proceed to prove this for both the efficient network structures that emerge corresponding to the cases when $2(\delta - \delta^2) < c < 2\delta + (n-2)\delta^2$ and $c < 2(\delta - \delta^2)$.

Case 1: When $2(\delta - \delta^2) < c < 2\delta + (n-2)\delta^2$, consider a strategy profile $s = (s_1, s_2, \ldots, s_n)$ that leads to a star network. The strategy profile is such that peripheral nodes form links to the center node and the center node does not form any link. To be more specific, let node 1 be the center node. The strategy profile s is such that $s_1 = \phi$ and $s_j = \{1\}\ \forall j \in \{2, 3, \ldots, n\}$. The utility of node 1 (i.e. center node) is $u_1(s) = (n-1)\delta$. Since node 1 is already directly connected to all the remaining nodes, it does not change its strategy. The utility of any peripheral node j under the strategy profile s is $u_j(s) = \delta - c + (n-2)\delta^2$. Let s' be a new strategy profile in which node j changes its strategy by forming the link to a peripheral node k instead of node 1. Now the utility of node j is $u_j(s') = \delta - c + \delta^2 + (n-3)\delta^3 < u_j(s)$. So it does change its strategy. Hence the strategy profile s is in Nash equilibrium.

Case 2: When $c < 2(\delta - \delta^2)$, consider a strategy profile $s = (s_1, s_2, \ldots, s_n)$ that leads to a wheel of length 3 with a local star (WS_3) consisting of all nodes. Let node 1 be the center node. To be more specific, the strategy profile s is such that $s_1 = \{2\},\ s_2 = \{3\}$, and $s_j = \{1\}, \ \ \forall j \in \{3, \ldots, n\}$. The utility of node 1 (i.e. center node) is $u_1(s) = (n-1)\delta - c$. Let s' be a new strategy profile in which only node 1 changes its strategy by forming the link to a peripheral node $j \neq 2$. Now the utility of node 1 is $u_1(s') = (n-1)\delta - c = u_1(s)$. As there is no benefit to move to s', it does not change its strategy. Also note that if s'' is another strategy profile in which the center node does not form any link to other nodes, then its utility is $u_1(s'') = (n-2)\delta < u_1(s)$ since $c < \delta$. Hence the center node does not deviate from its specified in the strategy s.

In similar fashion, we can show that the utility of any peripheral node under the strategy profile s is at least the utility of any other strategy profile. Hence the strategy profile s is in Nash equilibrium. ∎

Lemma 3 shows that each efficient network is also stable. However, there exist stable networks that are not efficient. For example, when $c < \delta$ and $k = 1$, consider the cycle network. It is straightforward to check that the cycle network is in Nash equilibrium. However note that it is not a efficient network. We note this observation in the following way.

Corollary 1: When $c < \delta$ and each node can form at most one link, every efficient network is also a stable network. But there are stable networks that are not efficient.

D. *Scenario 2: $k = 2$*

Let us consider a network g with less than $(2n-2)$ links. As each node can form at most 2 links, there must exist one node i in g with at most one link. Since $c < \delta$, such a node i forms a new link to improve its utility. This in turn increases the value of the network g. For this reason, our focus is only on those networks in which the number of edges is at least $(2n-2)$.

1) Analysis with $(2n-2)$ Links: The following lemma shows that the most efficient network among the networks with $(2n-2)$ links is also an equilibrium network.

Lemma 4: The efficient network among the networks with $(2n-2)$ links is a wheel network if
$$c < min\left(\delta - \delta^2,\ 2\delta + (n-4)\frac{\delta^2}{2}\right).$$
Moreover, it is a stable network.

Proof: To prove this, our approach is similar to the arguments presented in [9]. Any node can establish a new link in a network if the benefit due to the direct connection is greater than any indirect connection; that is, $(\delta - c) > \delta^2$ which implies that $c < (\delta - \delta^2)$.

Consider a network with n nodes and there must be at least $l \geq (n-1)$ links for that network to be connected. The value due to direct links is $l(2\delta - c)$ and the value due to indirect links is at most $(n(n-1) - 2l)\delta^2$. Hence the value of the component is at most

$$l(2\delta - c) + (n(n-1) - 2l)\delta^2 \quad (1)$$

If this network is a wheel network, then its value is

$$(2n-2)(2\delta - c) + (n-1)(n-4)\delta^2 \quad (2)$$

Now subtracting (2) from (1) yields

$$(l - (2n-2))(2\delta - c - 2\delta^2) \quad (3)$$

Note that $2\delta - c - 2\delta^2 > 0$ since it is given that $c < (\delta - \delta^2)$. When $l < (2n-2)$, then the value of equation (3) is negative and it is 0 when $l = (2n-2)$. A graph that is not a wheel must have a path of length at least 3 and it gets a value less than $2\delta^2$. Hence such a graph is not efficient. Further, it is easy to see that the value of a wheel network with n nodes is greater than the sum of the values of two wheel networks with n_1 and n_2 networks respectively such that $n = n_1 + n_2$. That is, the efficient network with $(2n-2)$ links is the wheel network and it encompasses all the nodes in the network.

The value of the wheel network is positive when

$$\begin{array}{c} (2n-2)(2\delta - c) + (n-1)(n-4)\delta^2 > 0 \\ \Rightarrow \quad c < 2\delta + (n-4)\frac{\delta^2}{2} \end{array}$$

Thus the wheel network is the efficient network among all network with $(2n-2)$ links if $c < min\left(\delta - \delta^2,\ 2\delta + (n-4)\frac{\delta^2}{2}\right)$.

We now proceed to prove that the wheel network is in equilibrium by exhibiting an equilibrium strategy profile. To be more specific, let node 1 be the center node with the rest being peripheral nodes. Consider a strategy profile $s = (s_1, s_2, s_3, \ldots, s_n)$ such that $s_1 = \phi,\ s_j = \{1, j+1\}\ \forall j \in \{2, 3, \ldots, n-1\}$, and $s_n = \{1, 2\}$. Clearly this strategy profile s results in a wheel network. It is simple to check that s is in equilibrium. This completes the proof of the lemma. ■

2) Analysis with $(2n-1)$ Links: Now we consider networks with $(2n-1)$ links. Note that the network $EWN1$ is an example for such a network.

Lemma 5: The efficient and stable network among the networks with $(2n-1)$ links is the $EWN1$ network when $c < min\left(\delta - \delta^2, 2\delta + \left[\frac{(n-1)(n-4)-2}{(2n-1)}\right]\delta^2\right)$.

Proof: Any node in a network forms a new link only if the benefit due to direct communication is greater than that of indirect communication; that is, $(\delta - c) > \delta^2$. From Lemma 4, we note that any network, call it $g1$, with $(2n-2)$ links that is not a wheel network must have a path of length at least 3 between a pair of nodes. Let $g2$ be the wheel network with $(2n-2)$ links. Consider the networks $g1$ and $g2$ and the values these networks is represented by $v(g1)$ and $v(g2)$ respectively. Since the shortest distance between any pair of nodes in $g2$ be at most 2 and there exists a path of length at least 3 between a pair of nodes in $g1$, we get that

$$v(g2) \geq v(g1) - 2\delta^3 + 2\delta^2 \quad (4)$$

We now add one link e to both $g1$ and $g2$ without violating the constraint that any node can form at most 2 links. We determine the values of $g1$ and $g2$ in the following.

The new link e can be added to $g1$ in two different ways. In the first case, the link e can be added between two nodes that are separated by a path of length 2. In this case the value of the new graph $g1' = g1 \cup \{e\}$ is given by

$$v(g1') = v(g1) + (2\delta - c) - 2\delta^2 \quad (5)$$

In the second case, the link e can be added to $g1'$ between two nodes that are separated a path of length at least 3. In this case the value of the new graph $g1'' = g1 \cup \{e\}$ is given by

$$v(g1'') = v(g1) + (2\delta - c) - 2\delta^3 \quad (6)$$

We now look at the graph $g2$. Note that all pairs of non-adjacent nodes in $g2$ are separated by paths of length 2 only. Hence the only way in which the new link e can be added to $g2$ is it must connect a pair of non-adjacent nodes. The value of new graph $g2' = g2 \cup \{e\}$ is given by

$$v(g2') = v(g2) + (2\delta - c) - 2\delta^2 \quad (7)$$

From the equations (5) and (7), we get that

$$\begin{array}{rcl} v(g2') - v(g1^{'}) & = & v(g2) - v(g1) \\ & \geq & 0 \text{ (from expression (4))} \end{array}$$

From the equations (6) and (7), we get that

$$\begin{array}{rcl} v(g2') - v(g1'') & = & v(g2) - v(g1) - 2\delta^2 + 2\delta^3 \\ & \geq & 0 \text{ (from expression (4))} \end{array}$$

We can conclude that the value of $g2'$ is greater than or equal to that of $g1'$ and $g1''$. Note that $g2'$ is nothing but $EWN1$. Hence $EWN1$ is an efficient network.

The value of the $EWN1$ network is positive when

$$\begin{array}{c} (2n-1)(2\delta - c) + [(n-1)(n-4) - 2]\,\delta^2 > 0 \\ \Rightarrow \quad c < 2\delta + \left[\frac{(n-1)(n-4)-2}{(2n-1)}\right]\delta^2 \end{array}$$

We now proceed to prove that the $EWN1$ network is in equilibrium by exhibiting an equilibrium strategy profile. To be more specific, let node 1 be the center node with the rest being peripheral nodes. Consider a strategy profile $s = (s_1, s_2, s_3, \ldots, s_n)$ such that $s_1 = \{i\}$ for some node i, $s_i = \{k, i+1\}$ where nodes i and k are not adjacent and $s_j = \{1, j+1\}$ $\forall j \in \{2, 3, \ldots, i-1, i+1, \ldots, n-1\}$, and $s_n = \{1, 2\}$. Clearly this strategy profile s results in a $EWN1$ network. It is simple to verify that s is in equilibrium. This completes the proof of the lemma.

■

3) Analysis with $2n$ Links: Now we consider networks with $2n$ links. Note that the network $EWN2$ is an example for such a network.

Lemma 6: The efficient and stable network among the networks with $2n$ links is the $EWN2$ network when

$$c < min\left(\delta - \delta^2, 2\delta + \left[\frac{(n-1)(n-4)-4}{2n}\right]\delta^2\right).$$

The proof of this lemma can be carried out on similar lines as that of Lemma 5.

4) A few Important Observations: We now look into the general case in the following lemma where we consider all networks having at most $2n$ links.

Lemma 7: Suppose a network can have at most $2n$ links and each node can form at most 2 links. Then

1) the efficient and stable network is the $EWN2$ network if $c < 2(\delta - \delta^2)$.
2) the efficient and stable network is the wheel network if $c > 2(\delta - \delta^2)$.

Proof: From the arguments in Section III-D, a network with less than $(2n-2)$ links cannot be an efficient network since $c < \delta$. Now from Lemma 4, the efficient network among the networks with $(2n-2)$ links is the wheel network. The value, call it $v(WHEEL)$, of the wheel network is

$$v(WHEEL) = (2n-2)(2\delta - c) + (n-1)(n-4)\delta^2$$

From Lemma 5, the efficient network among the networks with $(2n-1)$ links is the $EWN1$ network. The value, call it $v(EWN1)$, of the $EWN1$ network is

$$v(EWN1) = (2n-1)(2\delta - c) + [(n-1)(n-4) - 2]\,\delta^2$$

From Lemma 6, the efficient network among the networks with $2n$ links is the $EWN2$ network. The value, call it $v(EWN2)$, of the $EWN2$ network is

$$v(EWN2) = 2n(2\delta - c) + [(n-1)(n-4) - 4]\,\delta^2$$

Now the $EWN2$ network is efficient among all networks with at most $2n$ links if $v(EWN2) > v(EWN1)$ and $v(EWN2) > v(WHEEL)$. This happens only when $c < 2(\delta - \delta^2)$.

Now the wheel network is efficient among all networks with at most $2n$ links if $v(WHEEL) > v(EWN2)$ and $v(WHEEL) > v(EWN1)$. This happens only when $c > 2(\delta - \delta^2)$.

From Lemmas 4 and 6, we know that both the wheel and the $EWN2$ networks are stable.

■

We now make the following remark.

Corollary 2: When $c < \delta$ and each node can form at most two links, every efficient network is also a stable network.

IV. Conclusions and Future Work

In this paper, we studied the problem of formation of social networks when the nodes are resource constrained. We investigated the relationship between the efficiency and the stability of social networks. We formulated the network formation process as a strategic form game in which each node can form at most k links, where k is a given parameter. Then we showed interesting results when $k = 1$ and when $k = 2$.

It is important and interesting to study the structure of efficient networks that are stable when each node can establish $k > 2$ links. Laoutaris et.al. [10] attempted to study this problem by investigating the existence of Nash equilibrium in a different model. However, their study does not investigate efficiency or stability issues.

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