Optimal Mechanism Design for Multi Unit Combinatorial Auctions

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Agenda

- Introduction to Mechanism Design and Auctions
- Optimal Auctions
- An Optimal Multi-Unit Auction with Single Minded Bidders
- Summary and Future work
Motivation

- General situations motivated by the real world problems
General situations motivated by the real world problems

Combinatorial Procurement Auction

1 percent improvement amounts to USD 6 million
Need for Mechanism Design

- **Critical need:** We need true costs and capacities of the suppliers

- **Mechanism Design** comes into play

- Mechanism Design is an important tool in microeconomics and widely used in many other settings

- Nobel Prize in Economics Sciences, 2007: Hurwicz, Myerson, Maskin
Mechanisms used in the context of today’s are called **Optimal Auctions**

Mechanism design with heterogeneous objects (multi dimensional private information) is a formidable challenge

This talk: **Optimal Multi Unit Combinatorial Auction**
**Game Theory**: Analysis of strategic interaction among players
Mechanism Design

- **Game Theory**: Analysis of strategic interaction among players
- **Mechanism Design**: Reverse engineering of games
Mechanism Design

- **Game Theory**: Analysis of strategic interaction among players

- **Mechanism Design**: Reverse engineering of games

- **Mechanism Design** is the art of designing rules of a game to achieve a specific outcome in presence of *multiple self-interested agents*, each with *private information* about their preferences.
Mechanism Design Framework

\[ N = \{1, 2, \ldots, n\} \]
\[ N = \{1, 2, \ldots, n\} \quad \text{and} \quad \Theta_1, \ldots, \Theta_n \]
Mechanism Design Framework

\[ \mathcal{N} = \{1, 2, \ldots, n\} \]

\[ \Theta_1, \ldots, \Theta_n \]

\[ X : \text{Set of Outcomes} \]
Mechanism Design Framework

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\[ u_1, u_2, \ldots, u_n : X \times \Theta_i \rightarrow \mathbb{R} \]
Mechanism Design Framework

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\[ X : \text{Set of Outcomes} \]

\[ u_1, u_2, \ldots, u_n : \]

\[ X \times \Theta_i \rightarrow \mathbb{R} \]

SCF

\[ f : \Theta_1 \times \ldots \Theta_n \rightarrow X \]
Mechanism Design Framework

\[ u_i : X \times \Theta_i \rightarrow \mathbb{R} \]

\[ \Phi_1(\cdot) \]

\[ \theta_1 \]

\[ u_2 : X \times \Theta_2 \rightarrow \mathbb{R} \]

\[ \Phi_2(\cdot) \]

\[ \theta_2 \]

\[ \vdots \]

\[ u_n : X \times \Theta_n \rightarrow \mathbb{R} \]

\[ \Phi_n(\cdot) \]

\[ \theta_n \]

\[ x \in X \]

Policy Maker
**Incentive Compatibility**

**DSIC**
For all players, it is a dominant **strategy** to **reveal truth**, irrespective of the strategies of the other players.

**BIC**
For all players, it is a **best response** to **reveal truth**, if other players are revealing truth.
Space of Mechanisms
Space of Mechanisms
Space of Mechanisms
Space of Mechanisms

AE

GROVES

BIC

DSIC
Space of Mechanisms
Space of Mechanisms
Auctions

- **First Price Auction (FPA)** for selling a single item.

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  - The bidder with the highest bid wins.

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Auctions

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Auctions

- **First Price Auction (FPA)** for selling a single item.
  - The bidder with the highest bid wins.
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- **Second Price Auction (SPA)** for selling a single item.
  - The bidder with the highest bid wins.
  - She pays as much as the second highest bid.
- **Vickrey** ¹ showed: The truth revelation is dominant strategy in second price auction.

Optimal Auction

Myerson\(^2\): Introduced the notion of “Optimal auction”

Myerson\textsuperscript{2}: Introduced the notion of “Optimal auction”

- Maximizes revenue to the seller

Optimal Auction

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Optimal Auctions Beyond Myerson

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Optimal Multi Unit Combinatorial Auction in the Presence of Single Minded, Capacitated Bidders
Assumptions

1. The sellers are **single minded**
2. The sellers can collectively fulfill the demands specified by the buyer
3. The sellers are **capacitated**
4. The seller will **never inflate his capacity** (This is an important assumption)
5. All the participants are **rational and intelligent**
## Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>Set of items the buyer is interested in buying, ${1, 2, \ldots, m}$</td>
</tr>
<tr>
<td>$D_j$</td>
<td>Demand for item $j$, $j = \ldots m$</td>
</tr>
<tr>
<td>$N$</td>
<td>Set of sellers, ${1, 2, \ldots, n}$</td>
</tr>
<tr>
<td>$c_i$</td>
<td>True cost of production of one unit of bundle of interest to seller $i$, $c_i \in [c_i, \bar{c}_i]$</td>
</tr>
<tr>
<td>$q_i$</td>
<td>True capacity for bundle which seller $i$ can supply, $q_i \in [q_i, \bar{q}_i]$</td>
</tr>
<tr>
<td>$\hat{c}_i$</td>
<td>Reported cost by the seller $i$</td>
</tr>
<tr>
<td>$\hat{q}_i$</td>
<td>Reported capacity by the seller $i$</td>
</tr>
<tr>
<td>$b_i$</td>
<td>Bid of the seller $i$, $b_i = (\hat{c}_i, \hat{q}_i)$</td>
</tr>
<tr>
<td>$b$</td>
<td>Bid vector, $(b_1, b_2, \ldots, b_n)$</td>
</tr>
<tr>
<td>$b_{-i}$</td>
<td>Bid vector without the seller $i$, i.e. $(b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$</td>
</tr>
<tr>
<td>$t_i(b)$</td>
<td>Payment to the seller $i$ when submitted bid vector is $b$</td>
</tr>
<tr>
<td>$T_i(b_i)$</td>
<td>Expected payment to the seller $i$ when he submits bid $b_i$. Expectation is taken over all possible values of $b_{-i}$</td>
</tr>
<tr>
<td>Notation</td>
<td>Description</td>
</tr>
<tr>
<td>-------------------</td>
<td>-----------------------------------------------------------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$x_i = x_i(b)$</td>
<td>Quantity of the bundle to be procured from the seller $i$ when the bid vector is $b$</td>
</tr>
<tr>
<td>$X_i(b_i)$</td>
<td>Expected quantity of the bundle to be procured from the seller $i$ when he submits bid $b_i$. Expectation is taken over all possible values of $b_{-i}$</td>
</tr>
<tr>
<td>$f_i(c_i, q_i)$</td>
<td>Joint probability density function of $(c_i, q_i)$</td>
</tr>
<tr>
<td>$F_i(c_i, q_i)$</td>
<td>Cumulative distribution function of $f_i(c_i, q_i)$</td>
</tr>
<tr>
<td>$f_i(c_i</td>
<td>q_i)$</td>
</tr>
<tr>
<td>$F_i(c_i</td>
<td>q_i)$</td>
</tr>
<tr>
<td>$H_i(c_i, q_i)$</td>
<td>Virtual cost function for seller $i$, $H_i(c_i, q_i) = c_i + \frac{F_i(c_i</td>
</tr>
<tr>
<td>$\rho_i(b_i)$</td>
<td>Expected offered surplus to seller $i$, when his bid is $b_i$</td>
</tr>
</tbody>
</table>

**Table:** Notation
× sellers may not be willing to reveal their true types
Incentive Compatibility

- sellers may not be willing to reveal their true types
- offer them incentives for reporting true costs and capacities
sellers may not be willing to reveal their true types

offer them incentives for reporting true costs and capacities

we propose the following incentive structure, \( \forall i \in N \),

\[
\rho_i(b_i) = T_i(b_i) - \hat{c}_i X_i(b_i), \text{ where } b_i = (\hat{c}_i, \hat{q}_i)
\]
We proved,

**Theorem 1**

Any mechanism in the presence of single minded, capacitated sellers is BIC and IR iff \( \forall i \in N \)

1. \( \rho_i(b_i) = \rho_i(\bar{c}_i, \hat{q}_i) + \int_{\hat{c}_i}^{\bar{c}_i} X_i(t, \hat{q}_i) dt \)

2. \( \rho_i(b_i) \geq 0, \) and non-decreasing in \( \hat{q}_i \) \( \forall \hat{c}_i \in [c_i, \bar{c}_i] \)

3. The quantity which seller \( i \) is asked to supply, \( X_i(c_i, q_i) \) is non-increasing in \( c_i \) \( \forall q_i \in [q_i, \bar{q}_i] \).
An Optimal Auction

The buyer’s problem is to solve,

$$\min E_b \sum_{i=1}^n t_i(b) \quad \text{s.t.}$$

1. $$t_i(b) = \rho_i(b) + \hat{c}_i x_i(b)$$
2. Theorem 1 holds true.
3. She procures at least $$D_j$$ units of each item $$j$$. 
An Optimal Auction

The buyer’s problem is to solve,

\[ \min \ E_b \sum_{i=1}^{n} t_i(b) \quad \text{s.t.} \]

1. \( t_i(b) = \rho_i(b) + \hat{c}_i x_i(b) \)
2. Theorem 1 holds true.
3. She procures at least \( D_j \) units of each item \( j \).

Optimal Auction

\[ \min \int_{q}^{\bar{q}} \int_{c}^{\bar{c}} \left( \sum_{i=1}^{n} H_i(c_i, q_i) x_i(c_i, q_i) \right) f(c, q) dc \ dq \quad \text{s.t.} \]

1. \( \forall \ i \), \( X_i(c_i, q_i) \) is non-increasing in \( c_i, \forall \ q_i \).
2. The Buyer’s minimum requirement of each item is satisfied.
Regularity Assumption

\[ H_i(c_i, q_i) = c_i + \frac{F_i(c_i | q_i)}{f_i(c_i | q_i)} \]

is non-increasing in \( q_i \) and non-decreasing in \( c_i \).
An Optimal auction : Under regularity Assumption

The buyer’s optimal auction is,

$$\min \sum_{i=1}^{n} x_i H_i(c_i, q_i) \quad \text{subject to}$$

1. $0 \leq x_i \leq q_i$
2. Buyer’s demands are satisfied.

The buyer pays each seller $i$ the amount

$$t_i = c_i x_i^* + \int_{c_i}^{\bar{c}_i} x_i(t, q_i) dt$$

where $x_i^*$ is what agent $i$ has to supply after solving the above problem.

Note: This auction enjoys Dominant Strategy Incentive Compatibility.
Summary

We have seen,

- Necessary and sufficient conditions for BIC and individual rationality

- Characterization of an optimal multi unit combinatorial procurement auction in the presence of single minded capacitated bidders

- An optimal auction, for the same, which is dominant strategy incentive compatible if some regularity condition holds true

It is important to note that though in this we talk about reverse auction settings, the similar results hold true for forward auction settings as well.
Directions for future work,

- Relax the assumption of single minded bidders
- Multi-unit extensions
- Volume discounts
- Efficient auctions which are DSIC extension to Krishna’s work [1]


Introduced notion of incentive compatibility [4]


Groves mechanism are the only DSIC mechanisms which are allocatively efficient. [5].

E. Maskin [6] Pioneer of Implementation theory

On Optimal Auctions: [7, 1, 8, 9, 10, 11, 12]

For more about mechanism design [13, 14, 15, 16, 17]
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Game Theoretic Problems in Network Economics and Mechanism Design Solutions.

Questions?
Thank You!!!