AFFINE MAXIMIZERS IN DOMAINS WITH SELFISH VALUATIONS∗

Swaprava Nath1 and Arunava Sen1

1Indian Statistical Institute, New Delhi

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Abstract
We consider the domain of selfish and continuous preferences over a “rich” allocation space and show that onto, strategyproof and allocation non-bossy social choice functions are affine maximizers. Roberts (1979) proves this result for a finite set of alternatives and an unrestricted valuation space. In this paper, we show that in a sub-domain of the unrestricted valuations with the additional assumption of allocation non-bossiness, using the richness of the allocations, the strategyproof social choice functions can be shown to be affine maximizers. We provide an example to show that allocation non-bossiness is indeed critical for this result. This work shows that an affine maximizer result needs certain amount of richness split across valuations and allocations.

1 Introduction
Allocating divisible resources or tasks among competing agents is a classical economic problem. The proliferation of the Internet and the rapid development of computing power have created a marketplace for electronic resources. The consumers in these markets are typically individuals, small or medium businesses who pay the service providers for Internet connectivity. The frequencies of mobile telephony or 2G/3G bandwidth are scarce resources and the businesses and service providers often have different valuations for the resources which are private information. Designing mechanisms that reveal the agents’ demands truthfully is therefore an important problem. There is an extensive literature that deals with such applications (Yaïche et al., 2000; Sahasrabudhe and Kar, 2008; Wang et al., 2012).

A similar problem arises in the context of cloud computing and cloud storage. Many large companies need big computing resources to run their services smoothly and databases to store massive data. Maintaining such computing or data storage facilities is often expensive - these companies therefore outsource this service to specialized service providers in exchange for money. Bayrak et al. (2011) provides a survey on the economics of cloud computing.

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All these markets use money extensively for their services, and agents privately observe their valuations or the costs of their tasks. Following the vast literature in this area, we use quasi-linear utilities to model the payoffs. In addition since the valuations are private and selfish (an agent’s valuation is a function of her own resource share). However, the valuations are not assumed to be increasing in the amount of the resources consumed. The following example shows that this situation occurs often in practice. Consider an agent who requires a certain amount of resource (to run her business optimally) say \( \lambda \in [0, 1] \) fraction of the divisible resource. However, there is a resource maintenance cost that increases proportionally with the allocated resource. As long as her allocation is \( \delta(< \lambda) \), her returns are more than the cost and the valuation increases. But if she is allocated more than a share of \( \lambda \), the cost of maintaining the resource is more than the returns and her valuation decreases. So the valuation has a peak in the amount of the resource share. In a similar way we can construct examples where multiples of a particular resource share is desired by an agent and therefore the valuation can have multiple peaks.

The main result of this paper shows that the \textit{strategyproof} and \textit{allocation non-bossy} mechanisms in this model must belong to the well-known \textit{affine maximizer} class. An immediate consequence of this result is that the payments are of a specific functional form.

In the following two subsections, we present the related literature and a panoramic sketch of the proof respectively.

1.1 Relationship to the literature

Our paper is related to two strands of literature. The first strand deals with the results that characterize social choice functions (SCF) that are \textit{affine maximizers}. \(^1\) The original result appears in Roberts (1979) which considers a finite set of allocations \( A \) (with \(|A| \geq 3\)) and an unrestricted valuation domain. This result was generalized to a compact set of allocations and unrestricted continuous valuations in Carbajal et al. (2013). Both these results require “sufficient” richness in the valuation space. In particular, they require the valuation function to have externalities on the allocations of other agents that are not compatible with \textit{selfish} valuations.

If valuations are selfish and allocations are finite, then the class of strategyproof SCFs is typically larger than that of affine maximizers. Lavi et al. (2003) characterizes strategyproof SCFs for combinatorial auctions with the additional assumption of \textit{independence of irrelevant alternatives}. Mishra and Quadir (2014) show that implementable SCFs for single object auctions are \textit{generalized utility maximizers}. This is a larger class than that of affine maximizers - moreover, it remains larger even with the non-bossiness assumption. There are other characterization results for restricted domains (see, e.g., Maya and Nisan (2012) for the two player cake-cutting problem) that show that the class of strategyproof SCFs is larger than that of affine maximizers. Our result shows that in the restricted domain of selfish valuations, we can restore the affine maximizer result if the \textit{allocation space} is “rich” and the SCF is allocation non-bossy. In order to obtain an affine maximizer result, one needs a certain amount of richness, which, in this setting, is \textit{split} across valuations and allocations.

\(^1\)The term ‘affine maximizer’ was coined in Meyer-ter Vehn and Moldovanu (2002).
The other strand of literature to which our paper is related is that on the division model introduced in Sprumont (1991). As in the division model, we do not assume increasingness of valuations. The domain restriction in the division model is the single-peakedness of preferences. Ours is a multi-dimensional model where the single-peakedness restriction is replaced by monetary transfers in quasi-linear form. There is some recent work on multi-dimensional versions of the division model, e.g., Cho and Thomson (2013); Morimoto et al. (2013).

1.2 Brief overview of the proof

The affine maximizer theorem in Roberts (1979) has been revisited several times with different proofs (Dobzinski and Nisan, 2009; Lavi et al., 2009; Mishra and Sen, 2012; Carbajal et al., 2013). Our proof builds on the arguments in Roberts (1979), Lavi et al. (2009) and Carbajal et al. (2013). The first two papers consider finite alternative space, while the last considers an uncountable compact alternative space with continuous valuations. As we have remarked earlier, all these papers require richness in the valuation space that is ruled out by the selfish valuations assumption. A critical element of these proofs is the construction of a value difference set for any two distinct allocations which is then shown to be a half-space. In the selfish valuation domain, this idea cannot be applied directly since there can exist two distinct allocations \(x \) and \(y\) whose \(i\)-th components \(x_i\) and \(y_i\) are identical. The value difference set for these two alternatives will have zero at the \(i\)-th component for all vectors and will cease to have the half-space property.

In order to overcome this difficulty, we construct a subset of the allocation space called the distinct component set (DCS) that has distinct elements in each of the components. We show the existence of a maximal DCS and show that it is dense in the allocation space. Our proof proceeds by establishing the affine maximizer result on this maximal and dense DCS and then extending it to the full allocation space. The allocation non-bossiness assumption is used critically to establish the appropriate monotonicity condition required for the result.

The rest of the paper is organized as follows. We present the formal model in Section 2, state the main result in Section 3 and provide its proof in Section 4. We make a remark about the revenue equivalence result in Section 5. In Section 6, we discuss a more general result that can be proved using the same proof. We conclude the paper in Section 7.

2 The Model and Definitions

Let \(N = \{1, \ldots, n\}, n \geq 2\) be the set of agents. There is a set \(J = \{1, \ldots, m\}\) of perfectly divisible tasks (or objects) to be allocated among the agents. Let \(x_{ij} \in [0, 1], i \in N, j \in J\) be the fraction of task \(j\) assigned to agent \(i\). We assume that each task is fully allocated, i.e., \(\sum_{i \in N} x_{ij} = 1, \forall j \in J\). Let \(x_i \equiv (x_{i1}, \ldots, x_{im}) \in A_i \equiv [0,1]^m\) denote the fractions of the tasks allocated to agent \(i\). An allocation \(x \in A \equiv A_1 \times \cdots \times A_n \equiv [0,1]^{n \times m}\) is therefore a matrix where the rows and columns correspond to the agents and tasks respectively. The \((i,j)\)-th element of this matrix is \(x_{ij}\), as explained above. Let us call this allocation to be task sharing allocation.

We assume that the agents’ valuations are continuous and selfish, i.e., it depends only on the task portions assigned to that particular agent. Therefore, agent \(i\)’s valuation function is a continuous
map \( u_i : A_i \rightarrow \mathbb{R} \). We let \( U \) denote the set of all such valuations, and refer to it as the set of unrestricted selfish valuations.

In addition to the valuation that an agent derives from the tasks assigned to her, she can also be compensated with (or charged) money. Moreover, the aggregate utility function is quasi-linear in money, i.e., given by \( u_i(x_i) + p_i \), where \( p_i \) is agent \( i \)'s monetary compensation.

We assume that the agent valuations are private information and must be elicited. The goal of the paper is to identify collective outcomes that induce agents to reveal their private information truthfully. The following definitions are standard in mechanism design literature.

**Definition 1 (Social Choice Function)** A social choice function (SCF) \( F \) is a mapping from the set of valuation profiles to the set of alternatives, i.e., \( U^n \rightarrow A \).

**Definition 2 (Strategyproofness)** A SCF \( F \) is strategyproof if there exist transfers \( p_i : U^n \rightarrow \mathbb{R}, i \in N \), such that for all \( u = (u_i, u_{-i}) \in U^n \),

\[
    u_i(F(u_i, u_{-i})) + p_i(u_i, u_{-i}) \geq u_i(F(u'_i, u_{-i})) + p_i(u'_i, u_{-i}), \quad \forall u'_i \in U, i \in N.
\]

A strategyproof SCF induces truth-telling in dominant strategies.

The problem of characterizing strategyproof SCFs in allocation problems with selfish preferences presents special difficulties because agents can affect the allocations of other agents without affecting their own. An axiom that addresses this issue in a “minimal” way is that of allocation non-bossiness. The non-bossiness axiom was introduced in Satterthwaite and Sonnenschein (1981), and has been used widely in the literature on incentive compatibility.  

**Definition 3 (Allocation Non-Bossiness)** An SCF \( F \) satisfies allocation non-bossiness (ANB) if \( \forall i \in N, \forall u_{-i} \in U^{n-1} \) and \( \forall u_i, u'_i \in U \) with \( F_i(u_i, u_{-i}) = F_i(u'_i, u_{-i}) \) implies \( F(u_i, u_{-i}) = F(u'_i, u_{-i}) \), where \( F_i(\cdot) \) is the \( i \)-th component of \( F(\cdot) \).

Suppose an agent’s allocation is unaffected by a change in her valuation. According to ANB, this must not change the allocation of any other agent. A well-known fact about strategyproofness is that a change in an agent’s valuation that does not affect her allocation, does not also affect her payment. Therefore, the “if” part of non-bossiness together with strategyproofness ensures that the change in the agent’s valuation does not change her overall utility. In such a situation, the allocations of other agents is not allowed to change. We place no restrictions on the payments of the other agents, so that their utilities may change.

It is important to distinguish our version of non-bossiness from a related notion which we refer to as outcome non-bossiness. In a quasi-linear domain, an outcome comprises of the allocation and payments. An SCF is outcome non-bossy if the following holds: if an agent does not change her outcome by changing her own valuation, she does not change the outcomes of the other agents.

Outcome non-bossiness and strategyproofness imply allocation non-bossiness. Outcome non-bossiness is therefore a stronger requirement than allocation non-bossiness. In fact, affine maximizers (defined below), in general, do not satisfy outcome non-bossiness.

\(^2\)For matching, see e.g., (Svensson, 1999; Pápai, 2000), and for exchange economies, see e.g., (Goswami et al., 2013; Momi, 2013)
A salient SCF introduced in Roberts (1979) is the affine maximizer, defined as follows.

**Definition 4 (Affine Maximizer)** An SCF $F : U^n \rightarrow A$ is an affine maximizer if there exists $w_i \geq 0, i \in N$, not all zero, and a continuous function $\kappa : A \rightarrow \mathbb{R}$ such that,

$$F(u) \in \arg\max_{x \in A} \left( \sum_{i \in N} w_i u_i(x) + \kappa(x) \right).$$

The continuity of the valuations and the compactness of the set of allocations ensure that the affine maximizer SCF is well-defined.

### 3 Main Result

In this section, we present the central result of the paper that considers strategyproof SCFs in the domain of unrestricted, selfish valuations.

**Theorem 1 (Affine Maximizers for Selfish Valuations)** If $n \geq 3$, every onto, ANB and strategyproof SCF $F : U^n \rightarrow A$ is an affine maximizer. If $n = 2$, the result holds without the ANB assumption.

The selfish valuations domain is of particular economic interest and is a strict sub-domain of the domain used in Roberts (1979). It is well-known that restricting the domain typically expands the class of strategyproof SCFs because it limits the manipulation possibilities of an agent. However, our result shows that in this specific case, the class of strategyproof SCFs continue to be affine maximizers with the mild additional assumption of ANB for $n \geq 3$.

The unrestricted selfish valuation assumption ensure that onto affine maximizers exist. Next, we show that there exist affine maximizers that are onto, strategyproof and ANB.

Pick an arbitrary onto affine maximizer and a tie-breaking rule $t$, i.e., $t : A \rightarrow \mathbb{R}$ is an injective mapping. The affine maximizer together with the tie-breaking rule produces an SCF $F$ in a natural way: at every valuation profile the tie-breaking rule picks a unique allocation from the set of affine maximizers. We can use standard arguments in the literature to show that this SCF is strategyproof. In particular, the following payment rule implements this SCF. For every $i \in N$:

$$p_i(u_i, u_{-i}) = \begin{cases} \frac{1}{w_i} \left( \sum_{j \neq i} w_j u_j(F(u)) + \kappa(F(u)) + h_i(u_{-i}) \right), & w_i > 0 \\ 0, & w_i = 0 \end{cases}$$

We claim that $F$ is ANB.

Assume for contradiction that $F$ is bossy. Therefore, there exists some agent $i$ such that $x = (x_i, x_{-i}) = F(u_i, u_{-i}) \in A(u_i, u_{-i})$, where $A(u_i, u_{-i})$ is the set of affine maximizers when the valuation profile is $(u_i, u_{-i})$, and $y = (x_i, y_{-i}) = F(u'_i, u_{-i}) \in A(u'_i, u_{-i})$. Now, from the definition of affine maximizer when the valuation profile is $(u_i, u_{-i})$, we have:

$$w_i u_i(x_i) + \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \geq w_i u_i(x_i) + \sum_{j \neq i} w_j u_j(y_j) + \kappa(y)$$

$$\Rightarrow \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \geq \sum_{j \neq i} w_j u_j(y_j) + \kappa(y)$$

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Similarly, for valuation profile \((u'_i, u_{-i})\), we have:

\[
 w_i u'_i(x_i) + \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) \geq w_i u'_i(x_i) + \sum_{j \neq i} w_j u_j(x_j) + \kappa(x)
\]

\[
 \Rightarrow \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) \geq \sum_{j \neq i} w_j u_j(x_j) + \kappa(x)
\]

Hence, we get:

\[
 \sum_{j \neq i} w_j u_j(y_j) + \kappa(y) = \sum_{j \neq i} w_j u_j(x_j) + \kappa(x) \quad (3)
\]

Thus \(x, y \in \mathcal{A}(u'_i, u_{-i})\) and \(x, y \in \mathcal{A}(u_i, u_{-i})\). However, this contradicts the definition of the tie-breaking rule \(t\). If \(x\) is chosen over \(y\) at the valuation profile \((u_i, u_{-i})\), then \(y\) cannot be chosen over \(x\) at the valuation profile \((u'_i, u_{-i})\).

The ANB assumption is necessary for our result. We provide an example that shows that there are strategyproof and bossy SCFs that are not affine maximizers.

**Example 1 (Sequence Decisive Agent)** For simplicity, we illustrate a SCF \(F\) with 3 agents, \(N = \{1, 2, 3\}\) and one item \(m = 1\) (see Figure 1). The arguments can clearly be extended to a setting with arbitrary \(n\) and \(m\). Let \(t\) be an arbitrary real number referred to as the threshold. The SCF \(F\) is defined below.

For any valuations profile \((u_1, u_2, u_3)\), the SCF picks \(x_1^* \in \text{argmax}_{x_1 \in [0,1]} u_1(x_1)\). Ties are broken in favor of the largest value of \(x_1^*\). In order to determine the allocations of agents 2 and 3, we need to consider the two following cases.

Case 1: \(u_1(x_1^*) \leq t\): Then \(x_2^* \in \text{argmax}_{x_2 \in [0,1-x_1^*]} u_2(x_2)\) with ties broken as before, and \(x_3^* = 1 - x_1^* - x_2^*\).

Case 2: \(u_1(x_1^*) > t\): In this case, the role of agents 2 and 3 is reversed, i.e., \(x_3^* \in \text{argmax}_{x_3 \in [0,1-x_1^*]} u_3(x_3)\) with ties broken as before, and \(x_2^* = 1 - x_1^* - x_3^*\).

![Figure 1: An example to illustrate the necessity of ANB.](image)

Clearly \(F\) is onto. It is bossy because agent 1 can change the allocations of agents 2 and 3 without changing her own allocation. This is illustrated in Figure 1 - for the valuation profiles \((u_1, u_2, u_3)\) and \((u'_1, u_2, u_3)\), the allocations are \([0.5 0.3 0.2]^{\top}\) and \([0.5 0.2 0.3]^{\top}\) respectively.
Strategyproofness of $F$ follows from standard arguments. We claim that $F$ is not an affine maximizer. To see this, we repeat the arguments from Equations 2 to 3 with $x = [0.5 \ 0.3 \ 0.2]^{\top}$ and $y = [0.5 \ 0.2 \ 0.3]^{\top}$ to get,

$$w_2u_2(0.3) + w_3u_3(0.2) + \kappa([0.5 \ 0.3 \ 0.2]^{\top}) = w_2u_2(0.2) + w_3u_3(0.3) + \kappa([0.5 \ 0.2 \ 0.3]^{\top})$$

$$\Rightarrow \ \kappa([0.5 \ 0.3 \ 0.2]^{\top}) = \kappa([0.5 \ 0.2 \ 0.3]^{\top}) \text{ since } u_2(0.3) = u_2(0.2), u_3(0.2) = u_3(0.3).$$

Pick $u'_3$ which agrees with $u_3$ everywhere except $[0.15, 0.25]$ and is shown by the dashed line in the figure. By definition, $F(u_1, u_2, u'_3) = [0.5 \ 0.3 \ 0.2]^{\top}$. However,

$$w_1u_1(0.5) + w_2u_2(0.3) + w_3u'_3(0.2) + \kappa([0.5 \ 0.3 \ 0.2]^{\top})$$

$$< w_1u_1(0.5) + w_2u_2(0.2) + w_3u'_3(0.3) + \kappa([0.5 \ 0.2 \ 0.3]^{\top}),$$

since $u_2(0.2) = u_2(0.3) = 0.2$ but $0.2 = u'_3(0.3) > u'_3(0.2) = 0$ and $\kappa$ is same at these two allocations. Hence, $F$ is not an affine maximizer. $\Box$

4 Proof of the Theorem

In order to present the proof, we need to define an important property. Since allocations are decomposable into components for every agent and valuations are selfish, it is possible for the component of agent $i$ to be identical in two different allocations. To account for this possibility we need the following modification of the notion of positive association of differences (PAD) which appears in Roberts (1979).

**Definition 5 (PAD-DC)** An SCF $F$ satisfies positive association of differences for distinct components (PAD-DC) if $\forall u, u' \in U^n, \forall y \in A \setminus \{x\}$ and $i \in N$,

$$[F(u) = x \text{ and } u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i), \text{ for all } y_i \neq x_i] \Rightarrow [F(u') = x].$$

Let $x$ be an allocation and $u$ be a valuation profile such that $F(u) = x$. Such a profile exists because of the onto-ness of $F$. If we consider all other allocations in $A$, there will be an allocation $z \in A$ such that $x_j = z_j$ for some $j \in N$. Since valuations are selfish, $u'_j(x_j) - u_j(x_j) = u'_j(z_j) - u_j(z_j)$ and the conclusion $F(u') = x$ cannot be inferred from PAD. However, if $u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i), \forall y_i \neq x_i$, we can conclude $F(u') = x$ from PAD-DC. Of course, PAD-DC implies PAD. Therefore, PAD-DC is a stronger property than PAD.

The next result shows that PAD-DC is an implication of ANB and strategyproofness.

**Lemma 1** If $n \geq 3$, every ANB, strategyproof SCF satisfies PAD-DC. For $n = 2$, every strategyproof SCF satisfies PAD-DC.

**Proof:** We prove the lemma in three steps. Let $F$ be strategyproof and ANB.

**Step 1:** Pick any $i \in N$ and let $(u_i, u_{-i}), (u'_i, u_{-i}) \in U^n$ be such that $F(u) = x$, and suppose $\forall y \in A \setminus \{x\}, u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i)$, for all $y_i \neq x_i$. We claim $F_i(u'_i, u_{-i}) = x_i$. 


Suppose not, i.e., $F_i(u'_i, u_{-i}) = z_i \neq x_i$. By assumption, $u'_i(x_i) - u_i(x_i) > u'_i(z_i) - u_i(z_i)$. However, strategyproofness of $F$ implies $u'_i(x_i) - u_i(x_i) \leq u'_i(z_i) - u_i(z_i)$, which is a contradiction. Hence $F_i(u'_i, u_{-i}) = x_i$.

Step 2: Consider the case $n \geq 3$. Since Step 1 is true for every $i \in N, u_i, u'_i \in U, u_{-i} \in U^{n-1}$, ANB implies that $F(u'_i, u_{-i}) = x$. If $n = 2$, it is trivially true since there are only two components in each allocation $x_1$ and $x_2$, which are row vectors of length $m$ and their sums equal the ‘all-one’ vector of length $m$. Hence, if $x_1$ does not change, nor does $x_2$.

Step 3: We use these arguments for every player. Fix $u, u' \in U^n$ as in the definition of PAD-DC, i.e., $F(u) = x$ and $u'_i(x_i) - u_i(x_i) > u'_i(y_i) - u_i(y_i)$, $\forall y \in A \setminus \{x\}, y_i \neq x_i, \forall i \in N$. We have shown that $F(u_i, u_{-i}) = x$ implies $F(u'_i, u_{-i}) = x$. We repeat Steps 1 and 2 above for the transition from the value profile $(u'_i, u_{-i})$ to $(u'_i, u'_j, u_{-ij})$ and conclude that $F(u'_i, u'_j, u_{-ij}) = x$. Proceeding in this manner we conclude that $F(u') = x$, as needed. ■

The next claim will be used repeatedly in the proof. It is a counterpart of Claim 1 in Lavi et al. (2009). Their argument cannot be used directly because of the domain restriction used in this paper.

**Claim 1** Let $F$ satisfy PAD-DC, and fix $u, u' \in U^n$. Then,

$$[F(u) = x \text{ and } u'_i(x_i) - u'_i(y_i) > u_i(x_i) - u_i(y_i) \text{ whenever } y_i \neq x_i, \forall i \in N] \Rightarrow [F(u') \neq y]$$

**Proof:** Suppose not, i.e., $F(u') = y$. We will construct a valuation $u''_i \in U, i \in N$ such that applying PAD-DC for the transition $u' \rightarrow u''$ we will get $F(u'') = y$, while its application for the transition $u \rightarrow u''$ would imply $F(u'') = x$, leading to a contradiction.

Construct $u''$ to satisfy:

$$u''_i(y_i) - u'_i(y_i) > u''_i(z_i) - u'_i(z_i), \forall z_i \neq y_i, \forall i \in N \tag{4}$$

and

$$u''_i(x_i) - u_i(x_i) > u''_i(z_i) - u_i(z_i), \forall z_i \neq x_i, \forall i \in N. \tag{5}$$

Pick a pair of allocations $x$ and $y$ such that the LHS of the implication occurs, i.e., $F(u) = x$ and $u'_i(x_i) - u'_i(y_i) > u_i(x_i) - u_i(y_i)$ whenever $y_i \neq x_i, \forall i \in N$. A simple rearrangement of the terms yields, $\beta = u_i(x_i) - u'_i(x_i) < u_i(y_i) - u'_i(y_i) = \alpha$. Let $\delta = \alpha - \beta$ which is positive by assumption. The construction of $u''_i$ follows the approach in Lavi et al. (2009). However, the main difference is that the construction ensures continuity of $u''_i$ while satisfying Equations (4) and (5). In particular, $u''_i(x_i) - u'_i(x_i) = \beta + \delta/4$ and $u''_i(y_i) - u'_i(y_i) = \alpha - \delta/4$. We show that it is possible to construct such a $u''_i$. Figure 2 illustrates three representative cases that can be adapted to an

Let $p_i(.), i \in N$ be the payment function that makes $F$ strategyproof. The required inequality is obtained by summing the following inequalities which are consequences of strategyproofness:

$$u_i(x_i) + p_i(u_i, u_{-i}) \geq u_i(z_i) + p_i(u'_i, u_{-i}),$$

$$u'_i(z_i) + p_i(u'_i, u_{-i}) \geq u'_i(x_i) + p_i(u_i, u_{-i}).$$
One of the central ideas of our proof is to prove that $F$ is an affine maximizer on a suitably chosen dense subset of the allocation set $A$. The proof is completed by extending the result to the entire allocation space using the continuity of the affine maximizer.

**Definition 6 (Distinct Component Set)** A set $S \subseteq A$ is called a distinct component set (DCS) if for every distinct $x, y \in S$, $x_i \neq y_i$, $\forall i \in N$. 

Figure 2: Illustration for the proof of Claim 1.
We consider the DCSs that contain a specific allocation \( x \). Denote the set of all DCSs containing \( x \) by \( \mathcal{D}_x \). Our next result shows that a \textit{Maximal and Dense DCS} (MD-DCS) exists in \( \mathcal{D}_x \).

**Proposition 1** For every \( x \in A \), there exists a maximal element in \( \mathcal{D}_x \) which is dense in \( A \).

The proof of this proposition is deferred to the appendix.

Pick an arbitrary \( x \in A \) and an MD-DCS \( D_x \). All the following results until Claim 9 refer to this allocation and MD-DCS. Claims 2 to 10 have counterparts in Lavi et al. (2009) and Roberts (1979) and we follow their arguments closely. Therefore most of the details are omitted. At certain places new constructions are required because of the special nature of our domain. We point this out at the appropriate places.

The \textit{value difference set} for any pair of distinct allocations \( y, z \in D_x \) is defined as follows.

\[
P(y, z) = \{ \alpha \in \mathbb{R}^n : \exists u \in U^n \text{ s.t. } u(y) - u(z) = \alpha \text{ and } F(u) = y \}.
\]

The set \( P(y, z) \) is nonempty since \( F \) is onto. The next claim shows that the positive quadrant starting from a point in \( P(y, z) \) is always contained in \( P(y, z) \).

**Claim 2** If \( \alpha \in P(y, z) \), and \( \delta > 0 \in \mathbb{R}^n \), then \( \alpha + \delta \in P(y, z) \), for all distinct \( y, z \in D_x \).

**Claim 3** For every \( \alpha, \epsilon \in \mathbb{R}^n \), \( \epsilon > 0 \), and for all \( y, z \in D_x \),

(a) \( \alpha - \epsilon \in P(y, z) \Rightarrow -\alpha \notin P(z, y) \).

(b) \( \alpha \notin P(y, z) \Rightarrow -\alpha \in P(z, y) \).

**Proof sketch:** (a) Suppose, for contradiction, \(-\alpha \in P(z, y)\). Then, there exists \( u \in U^n \) such that \( u(z) - u(y) = -\alpha \) and \( F(u) = z \). Since \( \alpha - \epsilon \in P(y, z) \), there exists \( u' \) such that \( u'(y) - u'(z) = \alpha - \epsilon \) and \( F(u') = y \). But \( u'(y) - u'(z) < u(y) - u(z) \). This is a contradiction, since \( F(u) \neq z \) according to Claim 1.

(b) This proof is constructive and is inspired by the ‘flexibility’ condition of valuations in Carbajal et al. (2013).

Let \( B_y \) and \( B_z \) be disjoint balls with centers at \( y \) and \( z \) respectively. These exist since \( y \) and \( z \) do not match in any of their components. Let us choose a \( u \in U^n \) such that \( u(y) - u(z) = \alpha \) and \( u_i(y_i) - u_i(w_i) > u'_i(y_i) - u'_i(w_i) \), for all \( u, u' \in F^{-1}(y), w \in A \setminus \{y \cup B_z\} \), and \( y_i \neq w, i \in N \). We also want \( u \) to satisfy: \( u_i(z_i) - u_i(w_i) > u'_i(z_i) - u'_i(w_i) \), for all \( u, u' \in F^{-1}(z), w \in A \setminus \{z \cup B_y\} \), and \( z_i \neq w, i \in N \).

We claim that a continuous \( u \) can be constructed due to the assumptions on \( U \) and \( D_x \). Figure 3 illustrates the construction in the case when there is a single task. The general case involves a straightforward extension of the same idea to multiple dimensions.

We now argue that \( F(u) \in \{y, z\} \). If not, then \( F(u) \) either belongs to \( A \setminus \{y \cup B_z\} \) or \( A \setminus \{z \cup B_y\} \). In both cases, Claim 1 and the earlier inequalities imply \( F(u) \neq w \), which is a contradiction. Since we have shown \( F(u) \in \{y, z\} \) and \( u(y) - u(z) = \alpha \notin P(y, z) \), we have \( F(u) = z \). Since \( u(z) - u(y) = -\alpha \), it follows that \(-\alpha \in P(z, y) \). \( \blacksquare \)

The next result shows the relationship between the sum of the interiors of two value difference sets and the interior of a third.
Figure 3: Illustration for the proof of Claim 3(b).

**Claim 4** For every $\alpha, \beta, \epsilon^\alpha, \epsilon^\beta \in \mathbb{R}^n$, $\epsilon^\alpha, \epsilon^\beta > 0$, and $\forall y, z, w \in D_x$,

$$\alpha - \epsilon^\alpha \in P(y, z) \text{ and } \beta - \epsilon^\beta \in P(z, w) \Rightarrow \alpha + \beta - (\epsilon^\alpha + \epsilon^\beta)/2 \in P(y, w).$$

The proof of this claim proceeds in the same way as that in Claim 3(b).

The minimum translation factor is defined as follows:

$$\gamma(y, z) = \inf \{ p \in \mathbb{R} \mid p1 \in P(y, z) \}, \forall \text{ distinct } y, z \in D_x.$$

**Claim 5** For all distinct $y, z \in D_x$, the infimum $\gamma(y, z)$ exists in $\mathbb{R}$.

**Claim 6** For all $y, z, w \in D_x$, the following holds:

(a) $\gamma(y, z) = -\gamma(z, y)$.
(b) $\gamma(y, w) = \gamma(y, z) + \gamma(z, w)$.

Unlike the standard affine maximizer setting (Roberts, 1979; Lavi et al., 2009) in finite setting, we require $\gamma(\cdot, \cdot)$ to satisfy a continuity property.

**Claim 7** The minimum translation factor $\gamma(y, z)$ is continuous in both arguments.

*Proof sketch:* A small change in $z$ gives rise to a small change in $u(z)$ since $u$ is continuous. This in turn leads to a small change in $P(y, z)$ and $\gamma(y, z)$. Hence $\gamma(y, z)$ is continuous in the second argument. Since $\gamma(y, z) = -\gamma(z, y)$, it is continuous in the first argument as well. ■

Define the translated set $C(y, z) = P(y, z) - \gamma(y, z)1 = \{ \alpha - \gamma(y, z)1 \mid \alpha \in P(y, z) \}$. Denote the ‘interior’ of $C(y, z)$ by $\mathcal{C}(y, z)$, i.e.,

$$\mathcal{C}(y, z) = \{ \alpha \in C(y, z) \mid \alpha - \epsilon \in C(y, z), \text{ for some } \epsilon > 0 \}.$$
Claim 8 \( \hat{C}(y, z) = \hat{C}(w, l), \) for all \( y, z, w, l \in D_x, \) \( y \neq z \) and \( w \neq l. \)

Note that this result includes the cases, \( \hat{C}(y, z) = \hat{C}(w, z) = \hat{C}(w, y) = \hat{C}(z, y). \) Therefore, the Claim 8 holds even when \( y, z, w, l \) are not all distinct.

In view of this claim, we denote \( C_x = \hat{C}(y, z) = \hat{C}(w, l). \) Our next claim shows that this set is also independent of \( x. \)

Claim 9 \( C_x = C_y, \forall x, y \in A. \)

Proof: Construct two points in \( D_x \cap D_y. \) Pick the first point \( z_1 \) such that all its components are different from the corresponding components of both \( x \) and \( y. \) This is possible to pick since both \( x \) and \( y \) have finite number of components and both \( D_x \) and \( D_y \) are maximal sets. The second point \( z_2 \) is picked such that all its components are different from the corresponding components of both \( x, y \) and \( z_1. \) The argument of existence follows similarly. Therefore, we conclude that \( C_x = \hat{C}(z_1, z_2) = C_y, \) as needed.

In view of Claim 9, the subscript in \( C_x \) can be removed. The next claim shows that this set is convex.

Claim 10 \( C \) is convex.

Proof sketch: Pick distinct \( \alpha, \beta \in C. \) We will show \( (\alpha + \beta)/2 \in C. \) Since \( C \) is open, it follows that \( C \) is convex. Fix distinct \( y, z, w \in D. \) By definition, \( \alpha \in \hat{P}(y, z) - \gamma(y, z)1 \) and \( \beta \in \hat{P}(z, w) - \gamma(z, w)1. \) Therefore, \( \alpha + \beta \in (\hat{P}(y, z) + \hat{P}(z, w)) - (\gamma(y, z)1 + \gamma(z, w)1) = \hat{P}(y, w) - \gamma(y, w)1 \) (using Claims 4 and 6). Hence, \( \alpha + \beta \in C. \)

We show that if \( \alpha \in C, \) then \( \alpha/2 \in C. \) Suppose not, i.e., \( \alpha/2 \notin \text{int}(C(w, z)) \) for some \( w, z \in D_x \) and \( x \in A. \) It can be either \( \alpha/2 \notin C(w, z) \) or \( \alpha/2 \notin \text{bd}(C(w, z)) \) \footnote{\( \text{bd}(S) \) denotes the boundary of \( S. \)}. Therefore, for all \( \delta \geq 0, \delta \neq 0, \) we have \( \alpha/2 - \delta \notin C(w, z). \) An immediate consequence of Claim 3 is \( -\alpha/2 + \delta \in C(w, z). \) Since \( \alpha \in C, \) there exists \( e^\alpha > 0 \) such that \( \alpha - e^\alpha \in C(y, z) \) for some \( y, z \in D_x. \) Since \( -\alpha/2 + \delta \in C(w, z) \) and \( \delta \) is arbitrary, we can choose \( \delta = e^\alpha/4 \) so that \( -\alpha/2 + e^\alpha/2 - e^\alpha/4 \in C(z, w). \) Hence, we conclude from Claim 4 that \( \alpha + (-\alpha/2) - e^\alpha/8 \in C(y, w), \) i.e., \( \alpha/2 \in C. \) But this is a contradiction.

Since \( C = \hat{C}(y, z) = \hat{C}(z, y) \) is convex and \( \hat{C}(y, z) = (\text{cl}(-\hat{C}(z, y)))^c \) (Claim 3) \footnote{\( \text{cl}(S) \) and \( S^c \) denote the closure and complement of \( S \) respectively.}, it follows that \( C \) is a convex half-space. Moreover, \( 0 \notin C. \) Applying the separating hyperplane theorem, it follows that there exists \( w \in \mathbb{R}_{\geq 0} \setminus \{0\} \) such that, \( w^\top \alpha \geq 0, \) for all \( \alpha \in \text{cl}(C). \)

Construction of the \( \kappa(\cdot) \) function Fix an arbitrary \( x_0 \in A. \)

\[
\kappa(x) = \begin{cases} 
\gamma(x_0, x) \cdot w^\top 1 & \text{if } x_0 \in D_x \setminus \{x\} \\
\lim_{x_n \to x_0, x_n \in D_x} \gamma(x_n, x) \cdot w^\top 1 & \text{if } x_0 \notin D_x \\
0 & \text{if } x = x_0
\end{cases}
\]
This function is well defined since $\gamma(y, z)$ is continuous in both arguments (Claim 7) and $D_x$ is dense in $A$. Hence every allocation $x_0 \in A$ is either an element of $D_x$ or a limit point of $D_x$. Therefore, if $F(u) = x$, we have $w^\top (u(x) - u(y) - \gamma(x, y) \mathbf{1}) \geq 0$ for all $y \in D_x$. We consider two cases. If $x_0 \in D_x$,

$$w^\top (u(x) - u(y) - (\gamma(x, x_0) + \gamma(x_0, y)) \mathbf{1}) \geq 0$$

$$\Rightarrow w^\top u(x) + \gamma(x_0, x)w^\top \mathbf{1} \geq w^\top u(y) + \gamma(x_0, y)w^\top \mathbf{1}$$

$$\Rightarrow w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y).$$

If $x_0 \notin D_x$,

$$w^\top (u(x) - u(y) - (\gamma(x, x_n) + \gamma(x_n, y)) \mathbf{1}) \geq 0, \text{ for some } x_n \in D_x$$

$$\Rightarrow w^\top u(x) + \gamma(x_n, x)w^\top \mathbf{1} \geq w^\top u(y) + \gamma(x_n, y)w^\top \mathbf{1}$$

$$\Rightarrow w^\top u(x) + \lim_{x_n \to x_0, x_n \in D_x, \forall n} \gamma(x_n, x)w^\top \mathbf{1} \geq w^\top u(y) + \lim_{x_n \to x_0, x_n \in D_x, \forall n} \gamma(x_n, y)w^\top \mathbf{1}$$

$$\Rightarrow w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y).$$

Hence we have shown that if $F(u) = x$,

$$w^\top u(x) + \kappa(x) \geq w^\top u(y) + \kappa(y), \forall y \in D_x. \quad (7)$$

Since $D_x$ is dense in $A$, each allocation $z \in A$ is either a member of $D_x$ or its limit point. Claim 7 implies that $\kappa$ is a continuous function. Let $\{y_n\} \to z$ where $y_n \in D_x, \forall n$. Thus Equation (7) holds for every $y_n$. Taking limits we conclude that,

$$w^\top u(x) + \kappa(x) \geq w^\top u(z) + \kappa(z), \forall z \in A.$$

This concludes the proof. \hspace{1cm} Q.E.D.

5 Revenue Equivalence

In this section, we show that the SCF $F$ also satisfies revenue equivalence. This is an observation based on the definition of revenue equivalence and a standard result (Rockafellar, 1997). The following definition of revenue equivalence is standard in literature.

**Definition 7 (Revenue Equivalence)** An SCF $F$ satisfies revenue equivalence if for any two payment rules $p$ and $p'$ that make $F$ strategyproof, there exist functions $\alpha_i : U^{n-1} \to \mathbb{R}$, such that,

$$p_i(u_i, u_{-i}) = p'_i(u_i, u_{-i}) + \alpha_i(u_{-i}), \forall u_i \in U, \forall u_{-i} \in U^{n-1}, \forall i \in N.$$ 

This essentially ensures that any two payment rules that make $F$ strategyproof differ only by a factor which is independent of individual agents’ valuations. The following result shows that convexity and linearity of valuations is sufficient for a strategyproof SCF to satisfy revenue equivalence.
**Theorem 2 (Rockafellar (1997); Krishna and Maenner (2001))** If the type space is convex and the valuations are linear in type, then a strategyproof SCF satisfies revenue equivalence.

In our setting, since $U$ is the space of unrestricted selfish valuations, it is clearly convex. Also the types of the agents are their valuations, which implies, trivially, that the valuations are linear in its type. Hence, the above result on revenue equivalence holds for $F$ and we conclude that $F$ satisfies revenue equivalence. Therefore, the payment that implements $F$ must be of the form given in Equation (1).

6 A More General Result

Our affine maximizer result rests on the fact that the allocation space is rich enough to admit a maximal and dense subset for each allocation where selfish valuations provide all possible value differences. The following generalization therefore follows immediately.

**Theorem 3 (Affine Maximizers for Selfish Valuations)** Let the space of allocations $A$ be separable into components for each agent and compact. If there exists at least one collection of MD-DCSs $\{D_x \mid x \in A\}$ such that for all $x, y \in A$, $|D_x \cap D_y| \geq 2$, then an onto, ANB and strategyproof SCF $F : U^n \rightarrow A$ is an affine maximizer.

7 Conclusions

From the results of this paper, it appears that in order to prove that strategyproof SCFs are affine maximizers, perhaps a certain amount of richness is required, which may be split across the set of valuations and allocations. We considered a sub-domain of the unrestricted valuations, namely the domain of selfish valuations but a “rich” set of allocations to prove our affine maximizer result.

An interesting open question is whether our results extend to more restricted domains, e.g., domains with increasing valuations.

References


**Appendix**

**Proof of Proposition 1**

Define a partial order $\succsim$ on $\mathcal{D}_x$ as follows. For all $E_x, E'_x \in \mathcal{D}_x$, $E_x \succsim E'_x \iff E'_x \subseteq E_x$.

**Lemma 2** $(\mathcal{D}_x, \succsim)$ has a maximal element.

*Proof:* Consider an arbitrary chain, i.e., a linearly ordered subclass , in $(\mathcal{D}_x, \succsim)$, say $(\mathcal{B}, \succsim|_{\mathcal{B}})$. By definition, $\mathcal{B} \subseteq \mathcal{D}_x$ and $\succsim|_{\mathcal{B}}$ is a restriction of $\succsim$ to $\mathcal{B}$, which is complete and anti-symmetric. Consider $E^B = \bigcup_{E \in \mathcal{B}} E$.

We claim that $E^B \in \mathcal{D}_x$. Suppose not. Then $\exists x, y \in E^B$, $x \neq y$ such that $x$ and $y$ are same at least in one component. By definition of $E^B$ and considering the fact that $(\mathcal{B}, \succsim|_{\mathcal{B}})$ is a chain, it must be true that $\exists E^{x,y} \in \mathcal{B}$ such that $x, y \in E^{x,y}$. This contradicts the fact $\mathcal{B} \subseteq \mathcal{D}_x$.

Notice that $E^B \succsim E$, $\forall E \in \mathcal{B}$. Hence every chain in $(\mathcal{D}_x, \succsim)$ has an upper bound in $\mathcal{D}_x$. By Zorn’s lemma, $(\mathcal{D}_x, \succsim)$ has a maximal element.

**Lemma 3** There exists a $D \in \mathcal{D}_x$ such that $D$ is dense in $A$.

*Proof sketch:* The proof is constructive. We first prove a claim that is used in the construction. Define a grid as a collection of axis-parallel planes. Hence a grid splits the space $A$ into several cells. A open grid cell profile is the collection of the open cells induced by the grid (see Figure 4 for an example in two dimensions).

**Claim 11** For each finite open grid cell profile $G$ of $A$, there exists $C_G \subseteq A$ such that,

(i) each cell of $G$ has exactly one point of $C_G$,

(ii) $C_G$ is a DCS.
Figure 4: Illustration for the proof of Lemma 3.

Proof sketch: Define a random experiment which selects one point uniformly at random for every open cell in the cell profile $G$, independent from every other cell. It is clear that it is a measure zero event to pick two points which have at least one identical coordinate. Hence, there exists at least one configuration which is a DCS.

We start with $x$ and call $D_1 = \{x\}$. Draw axis parallel planes from $x$ so that it divides $A$ into a finite open grid cell profile $G_1$. Pick a DCS $C_{G_1}$. This is possible by the above claim. Note that $\{x\} \cup C_{G_1}$ is also a DCS. Let us call these set of points $D_2$. Now we draw axis parallel planes from each of the points in $D_2$, consider the corresponding finite grid cell profile $G_2$ and pick a DCS $C_{G_2}$ and construct $D_3$ in a similar way. We claim that $D := \bigcup_{i=1}^{\infty} D_i$ is dense in $A$. Clearly $D \in \mathcal{D}_x$ by construction. It is easy to show that for every $z \in A$ and $\epsilon > 0$, there exists a $n_\epsilon$, large enough, such that $\exists y \in D_{n_\epsilon}$ having $||y - z|| < \epsilon$. This concludes the proof.

Pick a $D \in \mathcal{D}_x$ dense in $A$. Define, $\chi_D = \{E_x \in \mathcal{D}_x \mid E_x \succ D\}$. It is clear that every element of $\chi_D$ is dense in $A$. Also $D \in \chi_D$, therefore, $\chi_D \neq \emptyset$.

**Lemma 4** $(\chi_D, \succ |_{\chi_D})$ has a maximal element.

The argument of the proof of this lemma is similar to Lemma 2. We show that this maximal element is a maximal element in the set of all DCSs.

**Lemma 5** Every maximal element of $(\chi_D, \succ |_{\chi_D})$ is a maximal element of $(\mathcal{D}_x, \succ)$.

**Proof:** Suppose not. Fix a maximal element $\theta_{\chi_D}$ of $(\chi_D, \succ |_{\chi_D})$ that is not maximal in $(\mathcal{D}_x, \succ)$. Therefore, there exists some $\theta \in \mathcal{D}_x$ such that $\theta \succ \theta_{\chi_D}$. By definition, $\theta_{\chi_D} \succ D$. By transitivity of order relation $\theta \succ D$, which implies $\theta \in \chi_D$. This contradicts the maximality of $\theta_{\chi_D}$ in $\chi_D$. ■
$(\chi_D, \preceq |_{\chi_D})$.

Hence, there exists a maximal element of $\mathcal{D}_x$ which is dense in $A$, as needed.